BIJECTIVE PROOF PROBLEMS

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The statements in each problem are to be proved combinatorially, in most cases by exhibiting an explicit bijection between two sets. Try to give the most elegant proof possible. Avoid induction, recurrences, generating functions, etc., if at all possible.

The following notation is used throughout for certain sets of numbers:

$\mathbb{N}$	nonnegative integers
$\mathbb{P}$	positive integers
$\mathbb{Z}$	integers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$[n]$	the set $\{1, 2, \ldots, n\}$ when $n \in \mathbb{N}$

We will (subjectively) indicate the difficulty level of each problem as follows:

- [1] easy
- [2] moderately difficult
- [3] difficult
- [u] unsolved
- [?] The result of the problem is known, but I am uncertain whether a combinatorial proof is known.
- [$*$] A combinatorial proof of the problem is not known. In all cases, the result of the problem is known.

Further gradations are indicated by + and –; e.g., [3–] is a little easier than [3]. In general, these difficulty ratings are based on the assumption that the solutions to the previous problems are known.

For those wanting to plunge immediately into serious research, the most interesting open bijections (but most of which are likely to be quite difficult) are Problems 27, 28, 59, 107, 143, 118, 123 (injection of the type described), 125, 140, 148, 151, 195, 198, 215, 216, 217, 226, 235, and 247.

CONTENTS

1. Elementary Combinatorics

1. [1] The number of subsets of an n-element set is 2^n.

2. [1] A *composition* of n is a sequence $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of positive integers such that $\sum \alpha_i = n$. The number of compositions of n is 2^{n-1}.

3. [2] The total number of parts of all compositions of n is equal to $(n+1)2^{n-2}$.

4. [2–] For $n \geq 2$, the number of compositions of n with an even number of even parts is equal to 2^{n-2}.

5. [2] Fix positive integers n and k. Find the number of k-tuples $(S_1, S_2, \dots, S_k)$ of subsets S_i of $\{1, 2, \dots, n\}$ subject to each of the following conditions *separately*, i.e., the three parts are independent problems (all with the same general method of solution).

   (a) $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k$

   (b) The S_i's are pairwise disjoint.

   (c) $S_1 \cap S_2 \cap \cdots \cap S_k = \emptyset$

6. [1] If S is an n-element set, then let $\binom{S}{k}$ denote the set of all k-element subsets of S. Let $\binom{n}{k} = \#\binom{S}{k}$, the number of k-subsets of an n-set. (Thus we are *defining* the binomial coefficient $\binom{n}{k}$ combinatorially when $n, k \in \mathbb{N}$.) Then

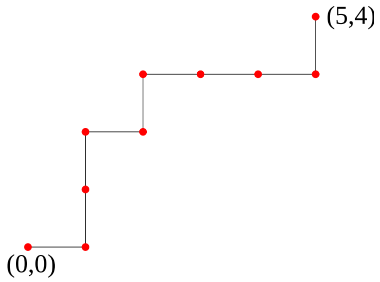
$$k!\binom{n}{k} = n(n-1)\cdots(n-k+1).$$

7. [1+] $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$. Here x and y are indeterminates and we define

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}.$$

   Note. Both sides are polynomials in x and y. If two polynomials $P(x,y)$ and $Q(x,y)$ agree for $x, y \in \mathbb{N}$ then they agree as polynomials. Hence it suffices to assume $x, y \in \mathbb{N}$.

8. [1] Let $m, n \geq 0$. How many lattice paths are there from $(0,0)$ to (m,n), if each step in the path is either $(1,0)$ or $(0,1)$? The figure below shows such a path from $(0,0)$ to $(5,4)$.



9. [1] For $n > 0$, $2\binom{2n-1}{n} = \binom{2n}{n}$.

10. [1+] For $n \geq 1$,
$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

11. [1+] For $n \geq 0$,
$$\sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k} = \binom{x+y}{n}. \tag{1}$$

12. [2–] For $n \geq 0$,
$$\sum_{k=0}^{n} \binom{x+k}{k} = \binom{x+n+1}{n}.$$

13. [3] For $n \geq 0$,
$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2(n-k)}{n-k} = 4^n.$$

14. [3–] We have
$$\sum_{i=0}^{m} \binom{x+y+i}{i}\binom{y}{a-i}\binom{x}{b-i} = \binom{x+a}{b}\binom{y+b}{a},$$
where $m = \min(a,b)$.

15. [3–] For $n \geq 0$,

$$\sum_{k=0}^{n} \binom{n}{k}^2 x^k = \sum_{j=0}^{n} \binom{n}{j} \binom{2n-j}{n} (x-1)^j.$$

16. [3–] Fix $n \geq 0$. Then

$$\sum_{i+j+k=n} \binom{i+j}{i} \binom{j+k}{j} \binom{k+i}{k} = \sum_{r=0}^{n} \binom{2r}{r}.$$

Here $i, j, k \in \mathbb{N}$.

17. [?] For $n \geq 0$,

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{n!^3}.$$

18. [3] Let $f(n)$ denote the number of subsets of $\mathbb{Z}/n\mathbb{Z}$ (the integers modulo n) whose elements sum to 0 (mod n) (including the empty set $\emptyset$). For instance, $f(5) = 8$, corresponding to $\emptyset$, $\{0\}$, $\{1, 4\}$, $\{0, 1, 4\}$, $\{2, 3\}$, $\{0, 2, 3\}$, $\{1, 2, 3, 4\}$, $\{0, 1, 2, 3, 4\}$. When n is odd, $f(n)$ is equal to the number of "necklaces" (up to cyclic rotation) with n beads, each bead colored white or black. For instance, when $n = 5$ the necklaces are (writing 0 for white and 1 for black) 00000, 00001, 00011, 00101, 00111, 01011, 01111, 11111. (This is easy if n is prime.)

19. [2–] How many $m \times n$ matrices of 0's and 1's are there, such that every row and column contains an odd number of 1's?

20. (a) [1–] Fix $k, n \geq 1$. The number of sequences $a_1 \cdots a_n$ such that $1 \leq a_i \leq k$ and $a_i \neq a_{i+1}$ for $1 \leq i < n$ is $k(k-1)^{n-1}$.

    (b) [2+] If in addition $a_1 \neq a_n$, then the number $g_k(n)$ of such sequences is

$$g_k(n) = (k-1)^n + (k-1)(-1)^n. \tag{2}$$

    Note. It's easy to prove bijectively that

$$g_k(n-1) + g_k(n) = k(k-1)^{n-1},$$

    from which (2) is easily deduced. I'm not sure, however, whether anyone has given a *direct* bijective proof of (2).

21. [2–] If p is prime and $a \in \mathbb{P}$, then $a^p - a$ is divisible by p. (A combinatorial proof would consist of exhibiting a set S with $a^p - a$ elements and a partition of S into pairwise disjoint subsets, each with p elements.)

22. (a) [2] Let p be a prime. Then $\binom{2p}{p} - 2$ is divisible by p^2.

    (b) $(*)$ In fact if $p > 3$, then $\binom{2p}{p} - 2$ is divisible by p^3.

23. [2–] If p is prime, then $(p-1)! + 1$ is divisible by p.

24. [1] A *multiset* M is, informally, a set with repeated elements, such as $\{1,1,1,2,4,4,4,5,5\}$, abbreviated $\{1^3, 2, 4^3, 5^2\}$. The number of appearances of i in M is called the *multiplicity* of i, denoted $\nu_M(i)$ or just $\nu(i)$. The definition of a *submultiset* N of M should be clear, viz., $\nu_N(i) \leq \nu_M(i)$ for all i. Let $M = \{1^{\nu_1}, 2^{\nu_2}, \ldots, k^{\nu_k}\}$. How many submultisets does M have?

25. [2] The *size* or *cardinality* of a multiset M, denoted $\#M$ or $|M|$, is its number of elements, counting repetitions. For instance, if

$$M = \{1,1,1,2,4,4,4,5,5\}$$

then $\#M = 9$. A multiset M is *on* a set S if every element of M is an element of S. Let $\left(\!\binom{n}{k}\!\right)$ denote the number of k-element multisets on an n-set, i.e., the number of ways of choosing, without regard to order, k elements from an n-element set if repetitions are allowed. Then

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}.$$

26. [2–] Fix $k, n \geq 0$. Find the number of solutions in nonnegative integers to

$$x_1 + x_2 + \cdots + x_k = n.$$

27. [*] Let $n \geq 2$ and $t \geq 0$. Let $f(n,t)$ be the number of sequences with n x's and $2t$ a_{ij}'s, where $1 \leq i < j \leq n$, such that each a_{ij} occurs between the ith x and the jth x in the sequence. (Thus the total number of terms in each sequence is $n + 2t\binom{n}{2}$.) Then

$$f(n,t) = \frac{(n + tn(n-1))!}{n!\, t!^n (2t)!^{\binom{n}{2}}} \prod_{j=1}^{n} \frac{((j-1)t)!^2 (jt)!}{(1+(n+j-2)t)!}.$$

NOTE. This problem a combinatorial formulation of a special case of the evaluation of a definite integral known as the *Selberg integral*. A combinatorial proof would be very interesting.

28. [*] A *binary de Bruijn sequence* of degree n is a binary sequence $a_1 a_2 \cdots a_{2^n}$ (so $a_i = 0$ or 1) such that all 2^n "circular factors" $a_i a_{i+1} \dots a_{i+n-1}$ (taking subscripts modulo 2^n) of length n are distinct. An example of such a sequence for $n = 3$ is 00010111. The number of binary de Bruijn sequences of degree n is $2^{2^{n-1}}$.

    NOTE. Note that $2^{2^{n-1}} = \sqrt{2^{2^n}}$. Hence if $\mathcal{B}_n$ denotes the set of all binary de Bruijn sequences of degree n and $\{0,1\}^{2^n}$ denotes the set of all binary sequences of length 2^n, then we want a bijection $\varphi : \mathcal{B}_n \times \mathcal{B}_n \to \{0,1\}^{2^n}$.

    NOTE. Binary de Bruijn sequences were defined and counted (nonbijectively) by Nicolaas Govert de Bruijn in 1946. It was then discovered in 1975 that this problem had been posed A. de Rivière and solved by C. Flye Sainte-Marie in 1894.

29. [3] Let α and β be two finite sequences of 1's and 2's. Define $\alpha < \beta$ if α can be obtained from β by a sequence of operations of the following types: changing a 2 to a 1, or deleting the last letter if it is a 1. Define $\alpha \prec \beta$ if α can be obtained from β by a sequence of operations of the following types: changing a 2 to a 1 if all letters preceding this 2 are also 2's, or deleting the first 1 (if it occurs). Given β and $k \geq 1$, let $A_k(\beta)$ be the number of sequences $\emptyset < \beta_1 < \beta_2 < \cdots < \beta_k = \beta$. Let $B_k(\beta)$ be the number of sequences $\emptyset \prec \beta_1 \prec \beta_2 \prec \cdots \prec \beta_k = \beta$. For instance, $A_3(22) = 7$, corresponding to $(\beta_1, \beta_2) = (2, 21)$, $(11, 21)$, $(1, 21)$, $(11, 12)$, $(1, 12)$, $(1, 11)$, $(1, 2)$. Similarly $B_3(22) = 7$, corresponding to $(\beta_1, \beta_2) = (2, 21)$, $(11, 21)$, $(1, 21)$, $(2, 12)$, $(1, 12)$, $(1, 11)$, $(1, 2)$. In general, $A_k(\beta) = B_k(\beta)$ for all k and β.

30. [1] The *Fibonacci numbers* F_n are defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. The number $f(n)$ of compositions of n with parts 1 and 2 is F_{n+1}. (There is at this point no set whose cardinality is known to be F_{n+1}, so you should simply verify that $f(n)$ satisfies the Fibonacci recurrence and has the right initial values.)

31. [2–] The number of compositions of n with all parts > 1 is F_{n-1}.

32. [2–] The number of compositions of n with odd parts is F_n.

33. [1+] How many subsets S of $[n]$ don't contain two consecutive integers?

34. [2–] How many binary sequences (i.e., sequences of 0's and 1's) $(\varepsilon_1, \ldots, \varepsilon_n)$ satisfy

$$\varepsilon_1 \leq \varepsilon_2 \geq \varepsilon_3 \leq \varepsilon_4 \geq \varepsilon_5 \leq \cdots ?$$

35. [2] Show that

$$\sum a_1 a_2 \cdots a_k = F_{2n},$$

where the sum is over all compositions $a_1 + a_2 + \cdots + a_k = n$.

36. [3–] Show that

$$\sum (2^{a_1 - 1} - 1) \cdots (2^{a_k - 1} - 1) = F_{2n-2},$$

where the sum is over all compositions $a_1 + a_2 + \cdots + a_k = n$.

37. [2] Show that

$$\sum 2^{\{\#i \,:\, a_i = 1\}} = F_{2n+1},$$

where the sum is over all compositions $a_1 + a_2 + \cdots + a_k = n$.

38. [2+] The number of sequences $(\delta_1, \delta_2, \ldots, \delta_n)$ of 0's, 1's, and 2's such that 0 is never immediately followed by a 1 is equal to F_{2n+2}.

39. [2?] Show that $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1}$.

40. [2–] Show that $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$.

41. [2] Continuing Exercise 5, fix positive integers n and k. Find the number of k-tuples $(S_1, S_2, \ldots, S_k)$ of subsets S_i of $\{1, 2, \ldots, n\}$ satisfying

$$S_1 \subseteq S_2 \supseteq S_3 \subseteq S_4 \supseteq S_5 \subseteq \cdots .$$

(The symbols $\subseteq$ and $\supseteq$ alternate.)

Bonus Chess Problem

(related to Problem 27)

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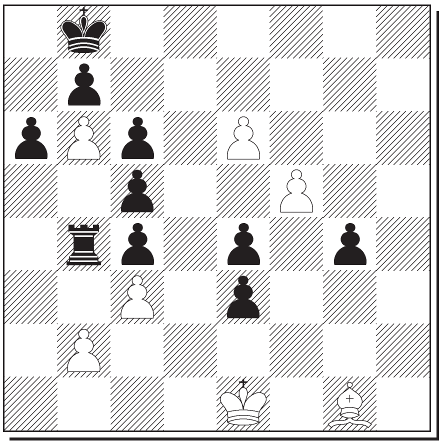
2004



Serieshelpmate in 14: how many solutions?

In a Serieshelpmate in n, Black makes n consecutive moves. White then makes one move, checkmating Black. Black may not check White (except possibly on his last move, if White then moves out of check) and may not move into check. White and Black are *cooperating* to achieve the goal of checkmate.

NOTE. For discussion of many of the chess problems given here, see

`www-math.mit.edu/~rstan/chess/queue.pdf`

2. Permutations

42. [1] In how many ways can n square envelopes of different sizes be arranged by inclusion? For instance, with six envelopes A, B, C, D, E, F (listed in decreasing order of size), one way of arranging them would be $F \in C \in B, E \in B, D \in A$, where $I \in J$ means that envelope I is contained in envelope J.

43. [2+] Let $f(n)$ be the number of sequences $a_1, \ldots, a_n$ of positive integers such that for each $k > 1$, k only occurs if $k-1$ occurs before the last occurrence of k. Then $f(n) = n!$. (For $n = 3$ the sequences are 111, 112, 121, 122, 212, 123.)

44. [2–] Let $w = a_1 a_2 \cdots a_n$ be a permutation of $1, 2, \ldots, n$, denoted $w \in \mathfrak{S}_n$. We can also regard w as the bijection $w : [n] \to [n]$ defined by $w(i) = a_i$. We say that i is a *fixed point* of w if $w(i) = i$ (or $a_i = i$). The total number of fixed points of all $w \in \mathfrak{S}_n$ is $n!$.

45. [2] An *inversion* of w is a pair (i, j) for which $i < j$ and $a_i > a_j$. Let $\mathrm{inv}(w)$ denote the number of inversions of w. Then

$$\sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

46. [1] For any $w \in \mathfrak{S}_n$, $\mathrm{inv}(w) = \mathrm{inv}(w^{-1})$.

47. [2–] How many permutations $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ have the property that for all $1 \le i < n$, the numbers appearing in w between i and $i+1$ (whether i is to the left or right of $i+1$) are all less than i? An example of such a permutation is 976412358.

48. [2–] How many permutations $a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ satisfy the following property: if $2 \le j \le n$, then $|a_i - a_j| = 1$ for some $1 \le i < j$? E.g., for $n = 3$ there are the four permutations 123, 213, 231, 321.

49. [2] A *derangement* is a permutation with no fixed points. Let $D(n)$ denote the number of derangments of $[n]$ (i.e., the number of $w \in \mathfrak{S}_n$ with no fixed points). (Set $D(0) = 1$.) Then

$$D(n) = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right). \tag{3}$$

NOTE. A rather complicated recursive bijection follows from a general technique for converting Inclusion-Exclusion arguments to bijective proofs. What is wanted, however, is a "direct" proof of the identity

$$D(n) + \frac{n!}{1!} + \frac{n!}{3!} + \cdots = n! + \frac{n!}{2!} + \frac{n!}{4!} + \cdots .$$

In other words, the number of ways to choose a permutation $w \in \mathfrak{S}_n$ and then choose an odd number of fixed points of w, or instead to choose a derangement in $\mathfrak{S}_n$, is equal to the number of ways to choose $w \in \mathfrak{S}_n$ and then choose an even number of fixed points of w.

50. [1] Show that

$$D(n) = (n-1)(D(n-1) + D(n-2)), \quad n \geq 1.$$

51. [3] Show that

$$D(n) = nD(n-1) + (-1)^n.$$

(Trivial from (3), but surprisingly tricky to do bijectively.)

52. [2] Let $m_1, \ldots, m_n \in \mathbb{N}$ and $\sum im_i = n$. The number of $w \in \mathfrak{S}_n$ whose disjoint cycle decomposition contains exactly m_i cycles of length i is equal to

$$\frac{n!}{1^{m_1}\, m_1!\, 2^{m_2}\, m_2! \cdots n^{m_n}\, m_n!}.$$

Note that, contrary to certain authors, we are including cycles of length one (fixed points).

53. [1+] A *fixed point free involution* in $\mathfrak{S}_{2n}$ is a permutation $w \in \mathfrak{S}_{2n}$ satisfying $w^2 = 1$ and $w(i) \neq i$ for all $i \in [2n]$. The number of fixed point free involutions in $\mathfrak{S}_{2n}$ is $(2n-1)!! := 1 \cdot 3 \cdot 5 \cdots (2n-1)$.

NOTE. This problem is a special case of Problem 52. For the present problem, however, give a factor-by-factor explanation of the product $1 \cdot 3 \cdot 5 \cdots (2n-1)$.

54. [3] If $X \subseteq \mathbb{P}$, then write $-X = \{-n : n \in X\}$. Let $g(n)$ be the number of ways to choose a subset X of $[n]$, and then choose fixed point free involutions π on $X \cup (-X)$ and $\bar{\pi}$ on $\bar{X} \cup (-\bar{X})$, where $\bar{X} = \{i \in [n] : i \notin X\}$. Then $g(n) = 2^n\, n!$.

55. [2–] Let $n \geq 2$. The number of permutations $w \in \mathfrak{S}_n$ with an even number of even cycles (in the disjoint cycle decomposition of w) is $n!/2$.

56. [2] Let $c(n,k)$ denote the number of $w \in \mathfrak{S}_n$ with k cycles (in the disjoint cycle decomposition of w). Then

$$\sum_{k=1}^{n} c(n,k)x^k = x(x+1)(x+2)\cdots(x+n-1).$$

Try to give *two* bijective proofs, viz., first letting $x \in \mathbb{P}$ and showing that both sides are equal as integers, and second by showing that the coefficients of x^k on both sides are equal.

57. [2] Let w be a random permutation of $1, 2, \ldots, n$ (chosen from the uniform distribution). Fix a positive integer $1 \leq k \leq n$. What is the probability that in the disjoint cycle decomposition of w, the length of the cycle containing 1 is k? In other words, what is the probability that k is the least positive integer for which $w^k(1) = 1$?

NOTE. Let p_{nk} be the desired probability. Then $p_{nk} = f_{nk}/n!$, where f_{nk} is the number of $w \in \mathfrak{S}_n$ for which the length of the cycle containing 1 is k. Hence one needs to determine the number f_{nk} by a bijective argument.

58. (a) [2] Let w be a random permutation of $1, 2, \ldots, n$ (chosen from the uniform distribution), $n \geq 2$. The probability that 1 and 2 are in the same cycle of w is $1/2$.

(b) [2+] Generalize (a) as follows. Let $2 \leq k \leq n$, and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ and $\sum \lambda_i = k$. Choose a random permutation $w \in \mathfrak{S}_n$. Let P_λ be the probability that $1, 2, \ldots, \lambda_1$ are in the same cycle C_1 of w, and $\lambda_1+1, \ldots, \lambda_1+\lambda_2$ are in the same cycle C_2 of w different from C_1, etc. Then

$$P_\lambda = \frac{(\lambda_1-1)!\cdots(\lambda_\ell-1)!}{k!}.$$

(c) [3–] Same as (b), except now we take w uniformly at random from the alternating group $\mathfrak{A}_n$. Let the resulting probability be Q_λ.

Then

$$Q_\lambda = \frac{(\lambda_1-1)!\cdots(\lambda_\ell-1)!}{(k-2)!}\left(\frac{1}{k(k-1)}+(-1)^{n-k}\frac{1}{n(n-1)}\right).$$

59. (a) [*] Let n be odd, and let u, v be random (independent, uniformly distributed) n-cycles in $\mathfrak{S}_n$. Then the probability that 1 and 2 are in the same cycle of the product uv is $1/2$.

    (b) [*] More generally, let $2 \leq k \leq n$ with $n-k$ odd. With u, v as above, the probability that $1, 2, \ldots, k$ are all in different cycles of uv is $1/k!$.

60. [3–] Let w be a random permutation of $1, 2, \ldots, n$ (chosen from the uniform distribution). For each cycle C of w, color all its elements red with probability $1/2$, and leave all its elements uncolored with probability $1/2$. The probability $P_k(n)$ that exactly k elements from $[n]$ are colored red is given by

$$P_k(n) = 4^{-n}\binom{2k}{k}\binom{2(n-k)}{n-k}.$$

Compare Problem 13.

61. [2+] A *record* (or *left-to-right maximum*) of a permutation $a_1a_2\cdots a_n$ is a term a_j such that $a_j > a_i$ for all $i < j$. The number of $w \in \mathfrak{S}_n$ with k records equals the number of $w \in \mathfrak{S}_n$ with k cycles.

62. [3] Let $a(n)$ be the number of permutations $w \in \mathfrak{S}_n$ that have a square root, i.e., there exists $u \in \mathfrak{S}_n$ satisfying $u^2 = w$. Then $a(2n+1) = (2n+1)a(2n)$.

63. [2+] Let $w = a_1 \cdots a_n \in \mathfrak{S}_n$. An *excedance* of w is a number i for which $a_i > i$. A *descent* of w is a number i for which $a_i > a_{i+1}$. The number of $w \in \mathfrak{S}_n$ with k excedances is equal to the number of $w \in \mathfrak{S}_n$ with k descents. (This number is denoted $A(n, k+1)$ and is called an *Eulerian number*.)

64. [2–] Continuing the previous problem, a *weak excedance* of w is a number i for which $a_i \geq i$. The number of $w \in \mathfrak{S}_n$ with k weak excedances is equal to $A(n, k)$ (the number of $w \in \mathfrak{S}_n$ with $k-1$ excedances).

65. [3–] Let $k \geq 0$. Then

$$\sum_{n\geq 0} n^k x^n = \frac{\sum_{k=1}^{n} A(n,k)x^n}{(1-x)^{k+1}}.$$

For instance,

$$\sum_{n\geq 0} n^3 x^n = \frac{x+4x^2+x^3}{(1-x)^4},$$

and there is one $w \in \mathfrak{S}_3$ with no descents, four with one descent, and one with two descents.

HINT. Given a permutation $a_1 a_2 \cdots a_n \in \mathfrak{S}_n$, consider all functions $f : [k] \to [n]$ satisfying: $f(a_1) \leq f(a_2) \leq \cdots \leq f(a_n)$ and $f(a_i) < f(a_{i+1})$ if $a_i > a_{i+1}$.

66. (a) [*] Given $m, n \geq 0$, define

$$C(m,n) = \frac{(2m)!(2n)!}{m!\,n!\,(m+n)!}.$$

Then $C(m,n) \in \mathbb{Z}$. (Note that $C(1,n) = 2C_n$, where C_n is a Catalan number.)

67. [3–] Let $f(n)$ be the number of ways to choose a subset $S \subseteq [n]$ and a permutation $w \in \mathfrak{S}_n$ such that $w(i) \notin S$ whenever $i \in S$. Then $f(n) = F_{n+1} n!$, where F_{n+1} denotes a Fibonacci number.

68. [1] Let $i_1, \ldots, i_k \in \mathbb{N}$, $\sum i_j = n$. The *multinomial coefficient* $\binom{n}{i_1,\ldots,i_k}$ is defined combinatorially to be the number of permutations of the multiset $\{1^{i_1}, \ldots, k^{i_k}\}$. For instance, $\binom{4}{1,2,1} = 12$, corresponding to the twelve permutations 1223, 1232, 1322, 2123, 2132, 2213, 2231, 2312, 2321, 3122, 3212, 3211. Then

$$\binom{n}{i_1,\ldots,i_k} = \frac{n!}{i_1! \cdots i_k!}.$$

69. [2] The *descent set* $D(w)$ of $w \in \mathfrak{S}_n$ is the set of descents of w. E.g., $D(47516823) = \{2,3,6\}$. Let $S = \{b_1, \ldots, b_{k-1}\} \subseteq [n-1]$, with $b_1 < b_2 < \cdots < b_{k-1}$. Let

$$\alpha_n(S) = \#\{w \in \mathfrak{S}_n : D(w) \subseteq S\}.$$

Then

$$\alpha_n(S) = \binom{n}{b_1, b_2 - b_1, b_3 - b_2, \ldots, b_{k-1} - b_{k-2}, n - b_{k-1}}.$$

70. [3–] The *major index* $\mathrm{maj}(w)$ of a permutation $w = a_1a_2\cdots a_n \in \mathfrak{S}_n$ is defined by

$$\mathrm{maj}(w) = \sum_{i\,:\,a_i > a_{i+1}} i = \sum_{i \in D(w)} i.$$

For instance, $\mathrm{maj}(47516823) = 2 + 3 + 6 = 11$. Then

$$\sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{maj}(w)}.$$

71. [3] Extending the previous problem, fix j, k, n. Then

$$\#\{w \in \mathfrak{S}_n \,:\, \mathrm{inv}(w) = j,\ \mathrm{maj}(w) = k\}$$
$$= \#\{w \in \mathfrak{S}_n \,:\, \mathrm{inv}(w) = k,\ \mathrm{maj}(w) = j\}.$$

Note. Problem 70 states that inv and maj are *equidistributed* on $\mathfrak{S}_n$, while Problem 71 states the stronger result that inv and maj have a *symmetric joint distribution* on $\mathfrak{S}_n$.

72. [2] A permutation $w = a_1a_2\cdots a_n \in \mathfrak{S}_n$ is *alternating* if $D(w) = \{1, 3, 5, \ldots\} \cap [n-1]$. In other words,

$$a_1 > a_2 < a_3 > a_4 < a_5 > \cdots.$$

Let E_n denote the number of alternating permutations in $\mathfrak{S}_n$. Then $E_0 = E_1 = 1$ and

$$2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}, \quad n \geq 1. \tag{4}$$

73. [2+] Show that

$$\sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x. \tag{5}$$

NOTE. It is not difficult to deduce this result from equation (4), but a combinatorial proof is wanted. This is quite a bit more difficult. Note that $\sec x$ is an even function of x and $\tan x$ is odd, so (5) is equivalent to

$$\begin{aligned} \sum_{n\geq 0} E_{2n}\frac{x^{2n}}{(2n)!} &= \sec x \\ \sum_{n\geq 0} E_{2n+1}\frac{x^{2n+1}}{(2n+1)!} &= \tan x. \end{aligned}$$

NOTE. We could actually use equation (5) to *define* $\tan x$ and $\sec x$ (and hence the other trigonometric functions in terms of these) combinatorially! The next two exercises deal with this subject of "combinatorial trigonometry."

74. [2+] Assuming (5), show that

$$1+\tan^2 x = \sec^2 x.$$

75. [2+] Assuming (5), show that

$$\tan(x+y) = \frac{\tan x + \tan y}{1-(\tan x)(\tan y)}.$$

76. [2] Let $k \geq 2$. The number of permutations $w \in \mathfrak{S}_n$ all of whose cycle lengths are divisible by k is given by

$$1^2\cdot 2\cdot 3\cdots(k-1)(k+1)^2(k+2)\cdots(2k-1)(2k+1)^2(2k+2)\cdots(n-1).$$

77. [3] Let $k \geq 2$. The number of permutations $w \in \mathfrak{S}_n$ none of whose cycle lengths is divisible by k is given by

$$1\cdot 2\cdots(k-1)^2(k+1)\cdots(2k-2)(2k-1)^2(2k+1)\cdots(n-1)n, \quad \text{if } k\nmid n$$

$$1\cdot 2\cdots(k-1)^2(k+1)\cdots(2k-2)(2k-1)^2(2k+1)\cdots(n-2)(n-1)^2, \quad \text{if } k|n.$$

78. [2] The number $u(n)$ of functions $f:[n]\to[n]$ satisfying $f^j=f^{j+1}$ for some $j\geq 1$ is given by $u(n)=(n+1)^{n-1}$, where f^i denotes iterated functional composition, e.g., $f^3(x)=f(f(f(x)))$. (Use Problem 128.)

79. [2] The number $g(n)$ of functions $f:[n]\to[n]$ satisfying $f=f^2$ is given by

$$h(n)=\sum_{i=1}^{n} i^{n-i}\binom{n}{i}.$$

80. [2+] The number $h(n)$ of functions $f:[n]\to[n]$ satisfying $f=f^j$ for some $j\geq 2$ is given by

$$h(n)=\sum_{i=1}^{n} i^{n-i}n(n-1)\cdots(n-i+1).$$

81. [3–] The number of pairs $(u,v)\in\mathfrak{S}_n^2$ such that $uv=vu$ is given by $p(n)n!$, where $p(n)$ denotes the number of partitions of n.

    NOTE (for those familiar with groups). This problem generalizes as follows. Let G be a finite group. The number of pairs $(u,v)\in G\times G$ such that $uv=vu$ is given by $k(G)\cdot|G|$, where $k(G)$ denotes the number of conjugacy classes of G. In this case a bijective proof is unknown (and probably impossible).

82. [2] The number of pairs $(u,v)\in\mathfrak{S}_n^2$ such that $u^2=v^2$ is given by $p(n)n!$ (as in the previous problem).

    NOTE. Again there is a generalization to arbitrary finite groups G. Namely, the number of pairs $(u,v)\in G\times G$ such that $uv=vu$ is given by $\iota(G)\cdot|G|$, where $\iota(G)$ denotes the number of *self-inverse* conjugacy classes K of G, i.e, if $w\in K$ then $w^{-1}\in K$.

83. [*] The number of triples $(u,v,w)\in\mathfrak{S}_n^3$ such that $u,v,$ and w are n-cycles and $uvw=1$ is equal to 0 if n is even (this part is easy), and to $2(n-1)!^2/(n+1)$ if n is odd.

84. [*] Let n be an odd positive integer. The number of ways to write the n-cycle $(1,2,\ldots,n)\in\mathfrak{S}_n$ in the form $uvu^{-1}v^{-1}$ $(u,v\in\mathfrak{S}_n)$ is equal to $2n\cdot n!/(n+1)$.

85. [?] Let $\kappa(w)$ denote the number of cycles of $w \in \mathfrak{S}_n$. Then

$$\sum_{w \in \mathfrak{S}_n} x^{\kappa(w\cdot(1,2,\ldots,n))} = \frac{1}{n(n+1)}((x+n)_{n+1} - (x)_{n+1}).$$

86. [3+] Let $a_{p,k}$ denote the number of fixed-point free involutions $w \in \mathfrak{S}_{2p}$ (i.e., the disjoint cycle decomposition of w consists of p 2-cycles) such that the permutation $w(1,2,\ldots,2p)$ has exactly k cycles. Then

$$\sum_{k\geq 1} a_{p,k}x^k = (1\cdot 3\cdot 5\cdots(2p-1))\sum_{k\geq 1} 2^{k-1}\binom{p}{k-1}\binom{x}{k}.$$

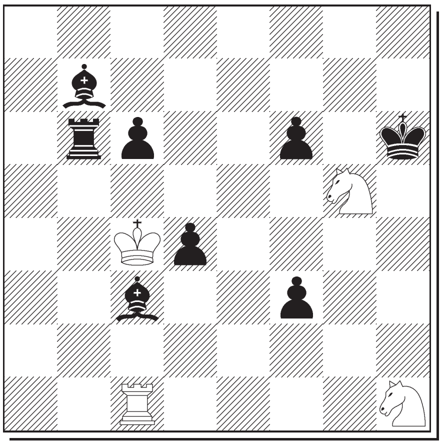
Note that Problem 175 gives the leading coefficient $a_{p,p+1}$.

Bonus Chess Problem

(related to Problem 72)

R. Stanley

2003

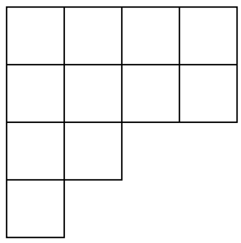


Serieshelpmate in 7: how many solutions?

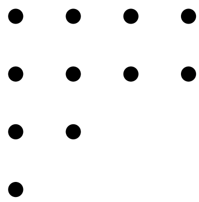
Can the theme of the above problem be extended another move or two?

3. Partitions

A *partition* λ of $n \geq 0$ (denoted $\lambda \vdash n$ or $|\lambda| = n$) is an integer sequence $(\lambda_1, \lambda_2, \dots)$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and $\sum \lambda_i = n$. Trailing 0's are often ignored, e.g., $(4,3,3,1,1)$ represents the same partition of 12 as $(4,3,3,1,1,0,0)$ or $(4,3,3,1,1,0,0,\dots)$. The terms $\lambda_i > 0$ are called the *parts* of λ. The *conjugate partition* to λ, denoted λ', has $\lambda_i - \lambda_{i+1}$ parts equal to i for all $i \geq 1$. The (Young) *diagram* of λ is a left-justified array of squares with λ_i squares in the ith row. For instance, the Young diagram of $(4,4,2,1)$ looks like



The Young diagram of λ' is the transpose of that of λ. Notation such as $u = (2,3) \in \lambda$ means that u is the square of the diagram of λ in the second row and third column. If dots are used instead of squares, then we obtain the *Ferrers diagram*. For instance, the Ferrers diagram of $(4,4,2,1)$ looks like



87. [1+] Let λ be a partition. Then

$$\sum_i (i-1)\lambda_i = \sum_i \binom{\lambda_i'}{2}.$$

88. [1+] Let λ be a partition. Then

$$\begin{aligned}
\sum_i \left\lceil \frac{\lambda_{2i-1}}{2} \right\rceil &= \sum_i \left\lceil \frac{\lambda'_{2i-1}}{2} \right\rceil \\
\sum_i \left\lfloor \frac{\lambda_{2i-1}}{2} \right\rfloor &= \sum_i \left\lceil \frac{\lambda'_{2i}}{2} \right\rceil \\
\sum_i \left\lfloor \frac{\lambda_{2i}}{2} \right\rfloor &= \sum_i \left\lfloor \frac{\lambda'_{2i}}{2} \right\rfloor .
\end{aligned}$$

89. [1] The number of partitions of n with largest part k equals the number of partitions of n with exactly k parts.

90. [2+] Fix $k \geq 1$. Let λ be a partition. Define $f_k(\lambda)$ to be the number of parts of λ equal to k, e.g., $f_3(8,5,5,3,3,3,3,2,1,1) = 4$. Define $g_k(\lambda)$ to be the number of integers i for which λ has at least k parts equal to i, e.g., $g_3(8,8,8,8,6,6,3,2,2,2,1) = 2$. Then

$$\sum_{\lambda \vdash n} f_k(\lambda) = \sum_{\lambda \vdash n} g_k(\lambda).$$

91. [2+] The total number of 2's in all partitions of n is equal to the total number of singletons in all partitions of $n-1$. A *singleton* is a part with multiplicity one. For instance, the partition $(8,8,7,5,4,4,4,2,2,1)$ has two 2's and three singletons.

92. [2–] The number of partitions of $n \geq 2$ into powers of 2 is even. For instance, when $n = 4$ there are the four partitions $4 = 2+2 = 2+1+1 = 1+1+1+1$.

93. [1] The number of partitions of n with k parts equals the number of partitions of $n + \binom{k}{2}$ with k distinct parts.

94. [2] The number of partitions of n with odd parts equals the number of partitions of n with distinct parts.

95. [2] The number of partitions of n for which no part occurs more than 9 times is equal to the number of partitions of n with no parts divisible by 10.

96. [2] Let $p(n)$ denote the number of partitions of n. The number of pairs (λ, μ), where $\lambda \vdash n$, $\mu \vdash n+1$, and the Young diagram of μ is obtained from that of λ by adding one square, is equal to $p(0)+p(1)+\cdots+p(n)$. (Set $p(0)=1$.)

97. [2] Let $\sigma(n)$ denote the sum of all (positive) divisors of $n \in \mathbb{P}$; e.g., $\sigma(12) = 1+2+3+4+6+12 = 28$. Let $p(n)$ denote the number of partitions of n (with $p(0)=1$). Then

$$n \cdot p(n) = \sum_{i=1}^{n} \sigma(i) p(n-i).$$

98. [2] The number of self-conjugate partitions of n equals the number of partitions of n into distinct odd parts.

99. [2+] Let $e(n)$, $o(n)$, and $k(n)$ denote, respectively, the number of partitions of n with an even number of even parts, with an odd number of even parts, and that are self-conjugate. Then $e(n)-o(n)=k(n)$.

100. [2] A *perfect partition* of $n \geq 1$ is a partition $\lambda \vdash n$ which "contains" precisely one partition of each positive integer $m \leq n$. In other words, regarding λ as the multiset of its parts, for each $m \leq n$ there is a unique submultiset of λ whose parts sum to m. The number of perfect partitions of n is equal to the number of *ordered* factorizations of $n+1$ into integers ≥ 2.

    Example. The perfect partitions of 5 are $(1,1,1,1,1)$, $(3,1,1)$, and $(2,2,1)$. The ordered factorizations of 6 are $6 = 2 \cdot 3 = 3 \cdot 2$.

101. (a) [3] The number of partitions of $5n+4$ is divisible by 5.

    (b) [*] (implies (a)) The number of 6-tuples $(\lambda^1, \dots, \lambda^6)$, where λ^1 is a partition of some integer $5k+4$, and the remaining λ^i's are all partitions of integers that are divisible by 5, such that $\sum |\lambda^i| = n$, is equal to five times the number of 6-tuples $(\mu^1, \dots, \mu^6)$, where each μ^i is a partition such that $\sum |\mu^i| = n$.

102. [3–] The number of incongruent triangles with integer sides and perimeter n is equal to the number of partitions of $n-3$ into parts equal to 2, 3, or 4. For example, there are three such triangles with perimeter 9, the side lengths being $(3,3,3), (2,3,4), (1,4,4)$. The corresponding partitions of 6 are 2+2+2=3+3=4+2.

103. [3] Let $f(n)$ be the number of partitions of n into an even number of parts, all distinct. Let $g(n)$ be the number of partitions of n into an odd number of parts, all distinct. For instance, $f(7) = 3$, corresponding to $6+1 = 5+2 = 4+3$, and $g(7) = 2$, corresponding to $7 = 4+2+1$. Then

$$f(n) - g(n) = \begin{cases} (-1)^k, & \text{if } n = k(3k \pm 1)/2 \text{ for some } k \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

NOTE. This result is usually stated in generating function form, viz.,

$$\prod_{n \geq 1} (1 - x^n) = 1 + \sum_{k \geq 1} (-1)^k \left(x^{k(3k-1)/2} + x^{k(3k+1)/2} \right),$$

and is known as *Euler's pentagonal number formula.*

104. [2] Let $f(n)$ (respectively, $g(n)$) be the number of partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ of n into distinct parts, such that the largest part λ_1 is even (respectively, odd). Then

$$f(n) - g(n) = \begin{cases} 1, & \text{if } n = k(3k+1)/2 \text{ for some } k \geq 0 \\ -1, & \text{if } n = k(3k-1)/2 \text{ for some } k \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

105. [3] For $n \in \mathbb{N}$ let $f(n)$ (respectively, $g(n)$) denote the number of partitions of n into distinct parts such that the smallest part is odd and with an even number (respectively, odd number) of even parts. Then

$$f(n) - g(n) = \begin{cases} 1, & \text{if } n \text{ is a square} \\ 0, & \text{otherwise.} \end{cases}$$

106. [3] Let $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$. Define

$$\begin{aligned} \alpha(\lambda) &= \sum_i \lceil \lambda_{2i-1}/2 \rceil \\ \beta(\lambda) &= \sum_i \lfloor \lambda_{2i-1}/2 \rfloor \\ \gamma(\lambda) &= \sum_i \lceil \lambda_{2i}/2 \rceil \\ \delta(\lambda) &= \sum_i \lfloor \lambda_{2i}/2 \rfloor. \end{aligned}$$

Let a, b, c, d be (commuting) indeterminates, and define

$$w(\lambda) = a^{\alpha(\lambda)} b^{\beta(\lambda)} c^{\gamma(\lambda)} d^{\delta(\lambda)}.$$

For instance, if $\lambda = (5, 4, 4, 3, 2)$ then $w(\lambda)$ is the product of the entries of the diagram

$$\begin{array}{ccccc} a & b & a & b & a \\ c & d & c & d & \\ a & b & a & b & \\ c & d & c & & \\ a & b & & & \end{array}$$

Show that

$$\sum_{\lambda} w(\lambda) = \prod_{j \geq 1} \frac{(1 + a^j b^{j-1} c^{j-1} d^{j-1})(1 + a^j b^j c^j d^{j-1})}{(1 - a^j b^j c^j d^j)(1 - a^j b^j c^{j-1} d^{j-1})(1 - a^j b^{j-1} c^j d^{j-1})},$$

where the sum on the left ranges over all partitions λ of all integers $n \geq 0$.

107. (a) (∗) The number of partitions of n into parts $\equiv \pm 1 \pmod 5$ is equal to the number of partitions of n whose parts differ by at least 2.

(b) (∗) The number of partitions of n into parts $\equiv \pm 2 \pmod 5$ is equal to the number of partitions of n whose parts differ by at least 2 and for which 1 is not a part.

NOTE. This is the combinatorial formulation of the famous *Rogers-Ramanujan identities*. Several bijective proofs are known, but none are really satisfactory. What is wanted is a "direct" bijection whose inverse is easy to describe.

108. [3] The number of partitions of n into parts $\equiv 1,\ 5$, or $6 \pmod 8$ is equal to the number of partitions into parts that differ by at least 2, and such that odd parts differ by at least 4.

109. [3] A *lecture hall partition* of length k is a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ (some of whose parts may be 0) satisfying

$$0 \leq \frac{\lambda_k}{1} \leq \frac{\lambda_{k-1}}{2} \leq \cdots \leq \frac{\lambda_1}{k}.$$

The number of lecture hall partitions of n of length k is equal to the number of partitions of n whose parts come from the set $1, 3, 5, \ldots, 2k-1$ (with repetitions allowed).

110. [3] The *Lucas numbers* L_n are defined by $L_1 = 1$, $L_2 = 3$, $L_{n+1} = L_n + L_{n-1}$ for $n \geq 2$. Let $f(n)$ be the number of partitions of n all of whose parts are Lucas numbers L_{2n+1} of odd index. For instance, $f(12) = 5$, corresponding to

$$\begin{array}{l} 1+1+1+1+1+1+1+1+1+1+1+1 \\ 4+1+1+1+1+1+1+1+1 \\ 4+4+1+1+1+1 \\ 4+4+4 \\ 11+1 \end{array}$$

Let $g(n)$ be the number of partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $\lambda_i/\lambda_{i+1} > \frac{1}{2}(3+\sqrt{5})$ whenever $\lambda_{i+1} > 0$. For instance, $g(12) = 5$, corresponding to

$$12, \quad 11+1, \quad 10+2, \quad 9+3, \quad 8+3+1.$$

Then $f(n) = g(n)$ for all $n \geq 1$.

111. [3–] Let $A(n)$ denote the number of partitions $(\lambda_1, \dots, \lambda_k) \vdash n$ such that $\lambda_k > 0$ and

$$\lambda_i > \lambda_{i+1} + \lambda_{i+2}, \quad 1 \leq i \leq k-1$$

(with $\lambda_{k+1} = 0$). Let $B(n)$ denote the number of partitions $(\mu_1, \dots, \mu_j) \vdash n$ such that

- Each μ_i is in the sequence $1, 2, 4, \dots, g_m, \dots$ defined by

$$g_1 = 1, \ g_2 = 2, \ g_m = g_{m-1} + g_{m-2} + 1 \text{ for } m \geq 3.$$

- If $\mu_1 = g_m$, then every element in $\{1, 2, 4, \dots, g_m\}$ appears at least once as a μ_i.

Then $A(n) = B(n)$ for all $n \geq 1$.

Example. $A(7) = 5$ because the relevant partitions are (7), $(6,1)$, $(5,2)$, $(4,3)$, $(4,2,1)$, and $B(7) = 5$ because the relevant partitions are $(4,2,1)$, $(2,2,2,1)$, $(2,2,1,1,1)$, $(2,1,1,1,1,1)$, $(1,1,1,1,1,1,1)$.

112. $(*)$ Let $S \subseteq \mathbb{P}$ and let $p(S,n)$ denote the number of partitions of n whose parts belong to S. Let

$$\begin{aligned} S &= \pm\{1,4,5,6,7,9,10,11,13,15,16,19 \pmod{40}\} \\ T &= \pm\{1,3,4,5,9,10,11,14,15,16,17,19 \pmod{40}\}, \end{aligned}$$

where

$$\pm\{a,b,\ldots \pmod m\} = \{n\in\mathbb{P} : n\equiv \pm a, \pm b, \ldots \pmod m\}.$$

Then $p(S,n) = p(T,n-1)$ for all $n\geq 1$.

NOTE. In principle the known proof of this result and of Problem 113 below can be converted into a complicated recursive bijection, as has been done for Problem 107. Just as for Probem 107, what is wanted is a "direct" bijection whose inverse is easy to describe.

113. [*] Let

$$\begin{aligned} S &= \pm\{1,4,5,6,7,9,11,13,16,21,23,28 \pmod{66}\} \\ T &= \pm\{1,4,5,6,7,9,11,14,16,17,27,29 \pmod{66}\}. \end{aligned}$$

Then $p(S,n) = p(T,n)$ for all $n\geq 1$ *except* $n=13$ (!).

114. [*] The number of partitions of $2n$ into distinct even parts equals the number of partitions if $2n+1$ into distinct odd parts, provided that all parts that are multiples of 7 are colored with one of two colors. (Two multiples of 7 that are different colors are regarded as different parts.) For instance, the partitions of 18 being counted are (18), $(16,2)$, $(14_1,4)$, $(14_2,4)$, $(12,6)$, $(12,4,2)$, $(10,8)$, $(10,6,2)$, $(8,6,4)$, while the partitions of 19 being counted are (19), $(15,3,1)$, $(11,7_1,1)$, $(11,7_2,1)$, $(11,5,3)$, $(9,7_1,3)$, $(9,7_2,3)$, $(7_1,7_2,5)$.

115. [2–] Prove the following identities by interpreting the coefficients in terms of partitions.

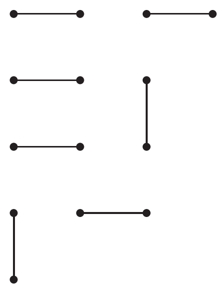
$$\begin{aligned}
\prod_{i\geq 1}\frac{1}{1-qx^i} &= \sum_{k\geq 0}\frac{x^k q^k}{(1-x)(1-x^2)\cdots(1-x^k)} \\
\prod_{i\geq 1}\frac{1}{1-qx^i} &= \sum_{k\geq 0}\frac{x^{k^2} q^k}{(1-x)\cdots(1-x^k)(1-qx)\cdots(1-qx^k)} \\
\prod_{i\geq 1}(1+qx^i) &= \sum_{k\geq 0}\frac{x^{\binom{k+1}{2}} q^k}{(1-x)(1-x^2)\cdots(1-x^k)} \\
\prod_{i\geq 1}(1+qx^{2i-1}) &= \sum_{k\geq 0}\frac{x^{k^2} q^k}{(1-x^2)(1-x^4)\cdots(1-x^{2k})}.
\end{aligned}$$

116. [3] Show that

$$\sum_{n=-\infty}^{\infty} x^n q^{n^2} = \prod_{k\geq 1}(1-q^{2k})(1+xq^{2k-1})(1+x^{-1}q^{2k-1}).$$

This famous result is *Jacobi's triple product identity*.

117. [3] Let $f(n)$ be the number of partitions of $2n$ whose Ferrers diagram can be covered by n edges, each connecting two adjacent dots. For instance, $(4,3,3,3,1)$ can be covered as follows:



Then $f(n)$ is equal to the number of ordered pairs (λ,μ) of partitions satisfying $|\lambda|+|\mu|=n$.

118. [3+] Given a partition λ and $u\in\lambda$, let $a(u)$ (called the *arm length* of u) denote the number of squares directly to the right of u (in the diagram of λ), counting λ itself exactly once. Similarly let $l(u)$ (called the *leg length* of u) denote the number of squares directly below u, counting u itelf once. Thus if $u=(i,j)$ then $a(u)=\lambda_i-j+1$ and $l(u)=\lambda'_j-i+1$. Define

$$\gamma(\lambda)=\#\{u\in\lambda\,:\,a(u)-l(u)=0 \text{ or } 1\}.$$

Then

$$\sum_{\lambda\vdash n} q^{\gamma(\lambda)}=\sum_{\lambda\vdash n} q^{\ell(\lambda)},$$

where $\ell(\lambda)$ denotes the length (number of parts) of λ.

119. [3–] If $0\leq k<\lfloor n/2\rfloor$, then $\binom{n}{k}\leq\binom{n}{k+1}$.

NOTE. To prove an inequality $a\leq b$ combinatorially, find sets A,B with $\#A=a$, $\#B=b$, and either an injection (one-to-one map) $f:A\to B$ or a surjection (onto map) $g:B\to A$.

120. [3–] Let $1 \le k \le n-1$. Then $\binom{n}{k}^2 \ge \binom{n}{k-1}\binom{n}{k+1}$. Note that this result is even stronger than Problem 119 above (assuming $\binom{n}{k} = \binom{n}{n-k}$) [why?].

121. [1] Let $p(j,k,n)$ denote the number of partitions of n with at most j parts and with largest part at most k. Then $p(j,k,n) = p(j,k,jk-n)$.

122. [3] Let $p(j,k,n)$ be as in the previous problem. A standard result in enumerative combinatorics states that

$$\sum_{n=0}^{jk} p(j,k,n)q^n = \begin{bmatrix} j+k \\ j \end{bmatrix},$$

where $\left[{m \atop i}\right]$ denotes the *q-binomial coefficient*:

$$\begin{bmatrix} m \\ i \end{bmatrix} = \frac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-i+1})}{(1-q^i)(1-q^{i-1})\cdots(1-q)}.$$

Prove this bijectively in the form

$$\frac{\sum_{n=0}^{jk} p(j,k,n)q^n}{(1-q^{j+k})(1-q^{j+k-1})\cdots(1-q^{k+1})} = \frac{1}{(1-q^j)(1-q^{j-1})\cdots(1-q)}.$$

123. [3] Continuing the previous problem, if $n < jk/2$ then $p(j,k,n) \le p(j,k,n+1)$.

Note. A (difficult) combinatorial proof is known. What is really wanted, however, is an injection $f : A_n \to A_{n+1}$, where A_m is the set of partitions counted by $p(j,k,m)$, such that for all $\lambda \in A_n$, $f(\lambda)$ is obtained from λ by adding 1 to a single part of λ. It is known that such an injection f exists, but no explicit description of f is known.

124. [1] Let $\bar{p}(k,n)$ denote the number of partitions of n into *distinct* parts, with largest part at most k. Then

$$\bar{p}(k,n) = \bar{p}(k, \binom{k+1}{2} - n).$$

Note. It is easy to see that

$$\sum_{n=0}^{\binom{k+1}{2}} \bar{p}(k,n)q^n = (1+q)(1+q^2)\cdots(1+q^k).$$

125. [*] Continuing the previous problem, if $n < \frac{1}{2}\binom{k+1}{2}$ then $\bar{p}(k,n) \leq \bar{p}(k,n+1)$.

NOTE. As in Problem 123 it would be best to give an injection $g : B_n \to B_{n+1}$, where B_m is the set of partitions counted by $\bar{p}(k,m)$, such that for all $\lambda \in B_n$, $f(\lambda)$ is obtained from λ by adding 1 to a single part of λ. It is known that such an injection g exists, but no explicit description of g is known. However, unlike Problem 123, *no* explicit injection $g : B_n \to B_{n+1}$ is known.

126. [2+] A *partition* π of a set S is a collection of nonempty pairwise disjoint subsets (called the *blocks* of π) of S whose union is S. Let $B(n)$ denote the number of partitions of an n-element set. $B(n)$ is called a *Bell number*. For instance, $B(3) = 5$, corresponding to the partitions (written in an obvious shorthand notation) 1-2-3, 12-3, 13-2, 1-23, 123. The number of partitions of $[n]$ for which no block contains two consecutive integers is $B(n-1)$.

127. [2] The number of permutations $w = a_1 \cdots a_n \in \mathfrak{S}_n$ such that for no $1 \leq i < j < n$ do we have $a_i < a_j < a_{j+1}$ is given by the Bell number $B(n)$. The same result holds if $a_i < a_j < a_{j+1}$ is replaced with $a_i < a_{j+1} < a_j$.

4. Trees

A *tree* T on $[n]$ is a graph with vertex set $[n]$ which is connected and contains no cycles. Equivalently, as is easy to see, T is connected and has $n-1$ edges. A *forest* is a graph for which every connected component is a tree. A *rooted tree* is a tree with a distinguished vertex u, called the *root*. If there are $t(n)$ trees on $[n]$ and $r(n)$ rooted trees, then $r(n) = nt(n)$ since there are n choices for the root u. A *planted forest* (sometimes called a *rooted forest*) is a graph for which every connected component is a rooted tree.

128. [3–] The number of trees $t(n)$ on $[n]$ is $t(n) = n^{n-2}$. Hence the number of rooted trees is $r(n) = n^{n-1}$.

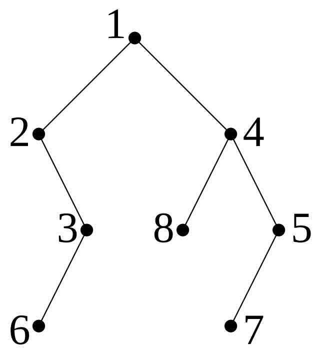
129. [1+] The number of planted forests on $[n]$ is $(n+1)^{n-1}$.

130. [2] Let $S \subseteq [n]$, $\#S = k$. The number $p_S(n)$ of planted forests on $[n]$ whose root set is S is given by
$$p_S(n) = kn^{n-k-1}.$$

131. [2] Given a planted forest F on $[n]$, let $\deg(i)$ be the *degree* (number of children of i). E.g., $\deg(i) = 0$ if and only if i is a leaf (endpoint) of F. If F has k components then it is easy to see that $\sum_i \deg(i) = n - k$. Given $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{N}^n$ with $\sum \delta_i = n - k$, let $N(\delta)$ denote the number of planted forests F on $[n]$ (necessarily with k components) such that $\deg(i) = \delta_i$ for $1 \leq i \leq n$. Then
$$N(\delta) = \binom{n-1}{k-1}\binom{n-k}{\delta_1, \ldots, \delta_n},$$
where $\binom{n-k}{\delta_1,\ldots,\delta_n}$ denotes a multinomial coefficient.

132. [3–] A *k-edge colored tree* is a tree whose edges are colored from a set of k colors such that any two edges with a common vertex have different colors. Show that the number $T_k(n)$ of k-edge colored trees on the vertex set $[n]$ is given by
$$T_k(n) = k(nk-n)(nk-n-1)\cdots(nk-2n+3) = k(n-2)!\binom{nk-n}{n-2}.$$

133. A *binary tree* is a rooted tree such that every vertex v has exactly two subtrees L_v, R_v, possibly empty, and the set $\{L_v, R_v\}$ is linearly ordered, say as (L_v, R_v). We call L_v the *left subtree* of v and draw it to the left of v. Similarly R_v is called the *right subtree* of v, etc. A binary tree on the vertex set $[n]$ is *increasing* if each vertex is smaller that its children. An example of such a tree is given by:



(a) [1+] The number of increasing binary trees on $[n]$ is $n!$.

(b) [2] The number of increasing binary trees on $[n]$ for which exactly k vertices have a left child is the Eulerian number $A(n, k+1)$.

134. An *increasing forest* is a planted forest on $[n]$ such that every vertex is smaller than its children.

(a) [1+] The number of increasing forests on $[n]$ is $n!$.

(b) [2] The number of increasing forests on $[n]$ with exactly k components is equal to the number of permutations $w \in \mathfrak{S}_n$ with k cycles.

(c) [2] The number of increasing forests on $[n]$ with exactly k endpoints is the Eulerian number $A(n, k)$.

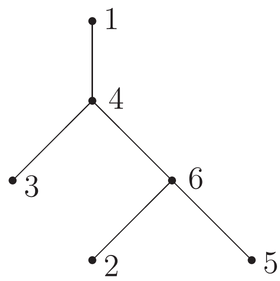
135. [2] Show that

$$\sum_{n\geq 0}(n+1)^n \frac{x^n}{n!} = \left(\sum_{n\geq 0} n^n \frac{x^n}{n!}\right)\left(\sum_{n\geq 0}(n+1)^{n-1}\frac{x^n}{n!}\right).$$

136. [2] Show that

$$\frac{1}{1-\sum_{n\geq 1} n^{n-1}\frac{x^n}{n!}} = \sum_{n\geq 0} n^n \frac{x^n}{n!}.$$

137. [3] Let τ be a rooted tree with vertex set $[n]$ and root 1. An *inversion* of τ is a pair (i,j) such that $1 < i < j$ and the unique path in τ from 1 to i passes through j. For instance, the tree below has the inversions $(3,4), (2,4), (2,6)$, and $(5,6)$.



Let $\mathrm{inv}(\tau)$ denote the number of inversions of τ. Define

$$I_n(t) = \sum_{\tau} t^{\mathrm{inv}(\tau)},$$

summed over all n^{n-2} trees on $[n]$ with root 1. For instance,

$$\begin{aligned}
I_1(t) &= 1\\
I_2(t) &= 1\\
I_3(t) &= 2+t\\
I_4(t) &= 6+6t+3t^2+t^3\\
I_5(t) &= 24+36t+30t^2+20t^3+10t^4+4t^5+t^6\\
I_6(t) &= 120+240t+270t^2+240t^3+180t^4+120t^5+70t^6+35t^7\\
&\quad +15t^8+5t^9+t^{10}.
\end{aligned}$$

Show that

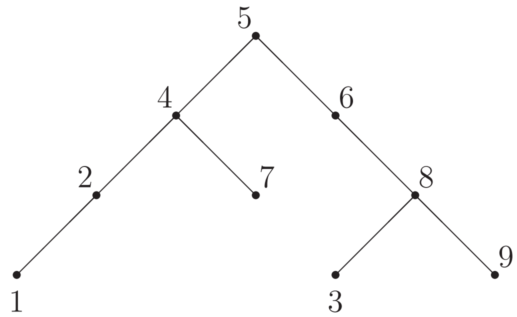
$$t^{n-1}I_n(1+t) = \sum_G t^{e(G)},$$

summed over all *connected* graphs G (without loops or multiple edges) on the vertex set $[n]$, where $e(G)$ is the number of edges of G.

138. [3] An *alternating tree* on $[n]$ is a tree with vertex set $[n]$ such that every vertex is either less than all its neighbors or greater than all its neighbors. Let $f(n)$ denote the number of alternating trees on $[n]$, so $f(1) = 1$, $f(2) = 1$, $f(3) = 2$, $f(4) = 7$, $f(5) = 36$, etc. Then

$$f(n+1) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (k+1)^{n-1}.$$

139. [3–] A *local binary search tree* is a binary tree, say with vertex set $[n]$, such that the left child of a vertex is smaller than its parent, and the right child of a vertex is larger than its parent. An example of such a tree is:



The number $f(n)$ of alternating trees on $[n]$ is equal to the number of local binary search trees on $[n]$.

140. [*] A *tournament* is a directed graph with no loops (edges from a vertex to itself) and with exactly one edge $u \to v$ or $v \to u$ between any two distinct vertices u, v. Thus the number of tournaments on $[n]$ (i.e., with vertex set $[n]$) is $2^{\binom{n}{2}}$. Write $C = (c_1, c_2, \dots, c_k)$ for the directed cycle with edges $c_1 \to c_2 \to \cdots \to c_k \to c_1$ in a tournament on $[n]$. Let $\operatorname{asc}(C)$ be the number of integers $1 \le i \le k$ for which $c_{i-1} < c_i$, and let $\operatorname{des}(C)$ be the number of integers $1 \le i \le k$ for which $c_{i-1} > c_i$, where by convention $c_0 = c_k$. We say that the cycle C is *ascending* if $\operatorname{asc}(C) \ge \operatorname{des}(C)$. For example, the cycles (a, b, c), (a, c, b, d), (a, b, d, c), and (a, c, d, b) are ascending, where $a < b < c < d$. A tournament T on $[n]$ is *semiacyclic* if it contains no ascending cycles, i.e, if for any directed cycle C in T we have $\operatorname{asc}(C) < \operatorname{des}(C)$. The number of semiacyclic tournaments on $[n]$ is equal to the number of alternating

trees on $[n]$. (This problem, usually stated in a different but equivalent form, has received a lot of attention. A solution would be well worth publishing.)

141. [2] An *edge-labelled alternating tree* is a tree, say with $n+1$ vertices, whose edges are labelled $1, 2, \ldots, n$ such that no path contains three consecutive edges whose labels are increasing. (The vertices are not labelled.) If $n > 1$, then the number of such trees is $n!/2$.

142. [2+] A *spanning tree* of a graph G is a subgraph of G which is a tree and which uses every vertex of G. The number of spanning trees of G is denoted $c(G)$ and is called the *complexity* of G. Thus Problem 128 is equivalent to the statement that $c(K_n) = n^{n-2}$, where K_n is the complete graph on n vertices (one edge between every two distinct vertices). The *complete bipartite graph* K_{mn} has vertex set $A \cup B$, where $\#A = m$ and $\#B = n$, with an edge between every vertex of A and every vertex of B (so mn edges in all). Then $c(K_{mn}) = m^{n-1}n^{m-1}$.

143. [*] The *n-cube* C_n (as a graph) is the graph with vertex set $\{0,1\}^n$ (i.e., all binary n-tuples), with an edge between u and v if they differ in exactly one coordinate. Thus C_n has 2^n vertices and $n2^{n-1}$ edges. Then
$$c(C_n) = 2^{2^n-n-1}\prod_{k=1}^{n} k^{\binom{n}{k}}.$$

144. [3–] A *parking function* of length n is a sequence $(a_1, \ldots, a_n) \in \mathbb{P}^n$ such that its increasing rearrangement $b_1 \leq b_2 \leq \cdots \leq b_n$ satisfies $b_i \leq i$. The parking functions of length three are 111, 112, 121, 211, 122, 212, 221, 113, 131, 311, 123, 132, 213, 231, 312, 321. The number of parking functions of length n is $(n+1)^{n-1}$.

145. [3] Let $\mathrm{PF}(n)$ denote the set of parking functions of length n. Then
$$\sum_{(a_1,\ldots,a_n)\in \mathrm{PF}(n)} q^{a_1+\cdots+a_n} = \sum_{\tau} q^{\binom{n+1}{2}-\mathrm{inv}(\tau)},$$
where τ ranges over trees on $[n+1]$ with root 1, and where $\mathrm{inv}(\tau)$ is defined in Problem 137.

146. [3–] A *valid* n-pair consists of a permutation $w = a_1 \cdots a_n \in \mathfrak{S}_n$, together with a collection I of pairs (i, j) such that

- If $(i,j) \in I$ then $1 \leq i < j \leq n$.
- If $(i,j) \in I$ then $a_i < a_j$.
- If $(i,j),(i',j') \in I$ and $\{i, i+1, \ldots, j\} \subseteq \{i', i'+1, \ldots, j')$, then $(i,j) = (i',j')$.

For example, let $n = 3$. For each $w \in \mathfrak{S}_3$ we put after it the number of sets I for which (w, I) is a valid 3-pair: 123 (5), 213 (3), 132 (3), 231 (2), 312 (2), 321 (1). The number of valid n-pairs is $(n+1)^{n-1}$.

147. (a) [3] Let T be a tournament on $[n]$, as defined in Problem 140. The *outdegree* of vertex i, denoted $\mathrm{outdeg}(i)$, is the number of edges pointing out of i, i.e., edges of the form $i \to j$. The *outdegree sequence* of T is defined by

$$\mathrm{out}(T) = (\mathrm{outdeg}(1), \ldots, \mathrm{outdeg}(n)).$$

For instance, there are eight tournaments on [3], but two have outdegree sequence $(1,1,1)$. The other six have distinct outdegree sequences, so the total number of distinct outdegree sequences of tournaments on [3] is 7. The total number of distinct outdegree sequences of tournaments on $[n]$ is equal to the number of forests on $[n]$.

(b) [3] More generally, let G be an (undirected) graph on $[n]$. An *orientation* $\mathfrak{o}$ of G is an assignment of a direction $u \to v$ or $v \to u$ to each edge uv of G. The *outdegree sequence* of $\mathfrak{o}$ is defined analogously to that of tournaments. The number of distinct outdegree sequences of orientations of G is equal to the number of spanning forests of G.

148. [*] Let G be a graph on $[n]$. The *degree* of vertex i, denoted $\deg(i)$, is the number of edges incident to i. The (ordered) *degree sequence* of G is the sequence $(\deg(1), \ldots, \deg(n))$. The number $f(n)$ of distinct degree sequences of simple (i.e., no loops or multiple edges) graphs on $[n]$ is given by

$$f(n) = \sum_Q \max\{1, 2^{d(Q)-1}\}, \tag{6}$$

where Q ranges over all graphs on $[n]$ for which every connected component is either a tree or has exactly one cycle, which is of odd length. Moreover, $d(Q)$ denotes the number of (odd) cycles in Q.

149. [3] The number of ways to write the cycle $(1,2,\ldots,n) \in \mathfrak{S}_{n+1}$ as a product of $n-1$ transpositions (the minimum possible) is n^{n-2}. (A *transposition* is a permutation $w \in \mathfrak{S}_n$ with one cycle of length two and $n-2$ fixed points.) For instance, the three ways to write $(1,2,3)$ are (multiplying right-to-left) $(1,2)(2,3)$, $(2,3)(1,3)$, and $(1,3)(1,2)$.

Note. It is not difficult to show bijectively that the number of ways to write *some* n-cycle as a product of $n-1$ transpositions is $(n-1)!\, n^{n-2}$, from which the above result follows by "symmetry." However, a direct bijection between factorizations of a *fixed* n-cycle such as $(1,2,\ldots,n)$ and labelled trees (say) is considerably more difficult.

150. [3–] The following four sets have an equal number of elements.

(a) *Ternary trees* with n vertices. A ternary tree is a rooted tree such that each vertex has three linearly ordered (say from left-to-right) subtrees, possibly empty. For instance, there are three ternary trees with two vertices.

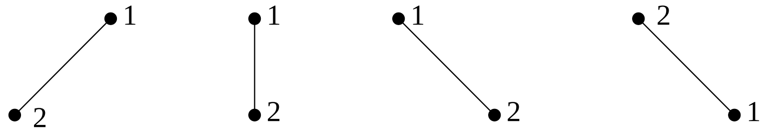
(b) *Noncrossing trees* on the vertex set $[n+1]$. A noncrossing tree T on a linearly ordered set S is a tree with vertex set S such that if $a<b<c<d$ in S, then not both ac and bd are edges of T.

(c) *Recursively labelled forests* on the vertex set $[n]$, i.e., a planted (or rooted) forest on $[n]$ such that the vertices of every subtree (i.e., of every vertex and all its descendants) is a set of consecutive integers.

(d) Equivalence classes of ways to write the cycle $(1,2,\ldots,n{+}1) \in \mathfrak{S}_n$ as a product of n transpositions (the minimum possible) such that two products are equivalent if they can be obtained from each other by successively interchanging consecutive commuting transpositions. (Two transpositions (i,j) and (h,k) commute if they have no letters in common.) Thus the three factorizations of $(1,2,3)$ are all inequivalent, while the factorization $(1,5)(2,4)(2,3)(1,4)$ of $(1,2,3,4,5)$ is equivalent to itself and $(2,4)(1,5)(2,3)(1,4)$, $(1,5)(2,4)(1,4)(2,3)$, $(2,4)(1,5)(1,4)(2,3)$, and $(2,4)(2,3)(1,5)(1,4)$. (Compare Problem 149.)

Note. The cardinality of the four sets above is equal to $\frac{1}{2n+1}\binom{3n}{n}$, a "ternary analogue" of the Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

151. [3] Let $f(n)$ be the number of labelled ternary trees on n vertices such that a left child or middle child has a larger label than its parent. Then $f(n) = C_n n!$, where C_n denotes a Catalan number. For instance, when $n = 2$ we are counting the four trees



152. [3+] Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition of n with $\lambda_\ell > 0$, and let w be a permutation of $1, 2, \ldots, n$ whose cycles have lengths $\lambda_1, \ldots, \lambda_\ell$. Let $f(\lambda)$ be the number of ways to write $w = t_1 t_2 \cdots t_k$ where the t_i's are transpositions that generate all of $\mathfrak{S}_n$, and where k is minimal with respect to the condition on the t_i's. (It is not hard to see that $k = n + \ell - 2$.) Show that

$$f(\lambda) = (n+\ell-2)!\, n^{\ell-3} \prod_{i=1}^{\ell} \frac{\lambda_i^{\lambda_i+1}}{\lambda_i!}.$$

NOTE. Suppose that $t_i = (a_i, b_i)$. Let G be the graph on $[n]$ with edges $a_i b_i$, $1 \le i \le k$. Then the statement that the t_i's generate $\mathfrak{S}_n$ is equivalent to the statement that G is connected.

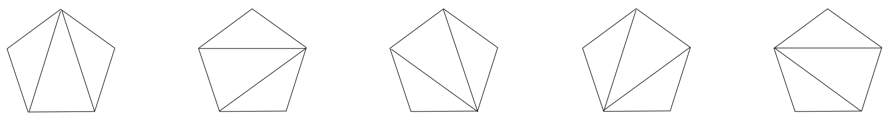
5. Catalan Numbers

Let us define the nth *Catalan number* C_n by

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 0. \tag{7}$$

Thus $(C_0, C_1, \ldots) = (1, 1, 2, 5, 14, 42, 132, 429, \ldots)$. There are a huge number of combinatorial interpretations of these numbers; 66 appear in Exercise 6.19 of R. Stanley, *Enumerative Combinatorics*, vol. 2 (available at `www-math.mit.edu/~rstan/ec`) and an addendum with many more interpretations may be found at the same website. We give here a subset of these interpretations that are the most fundamental or most interesting. Problem 164 is perhaps the easiest one to show bijectively is counted by (7). All your other proofs should be bijections with previously shown "Catalan sets." Each interpretation is illustrated by the case $n = 3$, which hopefully will make any undefined terms clear.

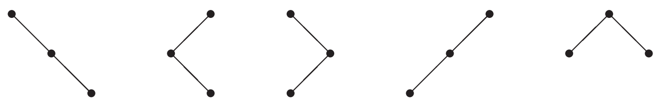
153. [2–] triangulations of a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors



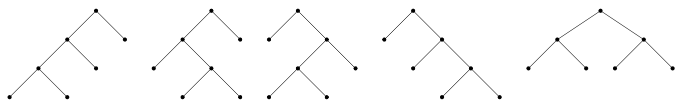
154. [1+] binary parenthesizations of a string of $n+1$ letters

$$(xx \cdot x)x \quad x(xx \cdot x) \quad (x \cdot xx)x \quad x(x \cdot xx) \quad xx \cdot xx$$

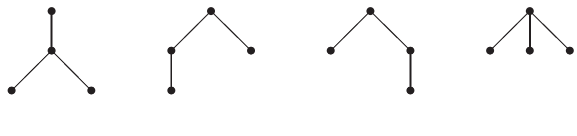
155. [1+] binary trees with n vertices



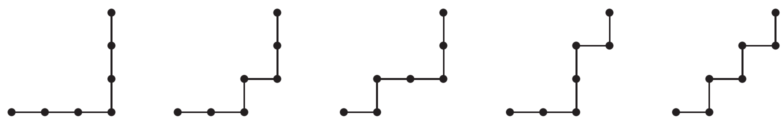
156. [1+] plane binary trees with $2n+1$ vertices (or $n+1$ endpoints) (A *plane binary tree* is a binary tree for which every vertex is either an endpoint or has two children.)



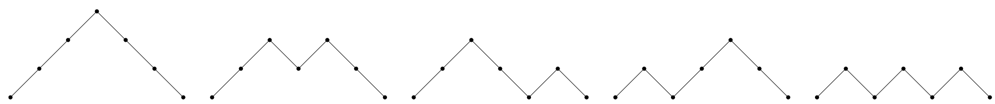
157. [2] plane trees with $n+1$ vertices (A *plane tree* is a rooted tree for which the subtrees of every vertex are linearly ordered from left to right.)



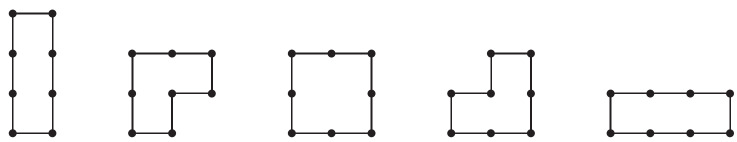
158. [1+] lattice paths from $(0,0)$ to (n,n) with steps $(0,1)$ or $(1,0)$, never rising above the line $y=x$



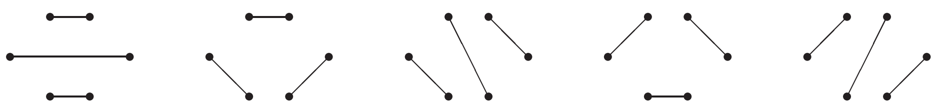
159. [1] Dyck paths from $(0,0)$ to $(2n,0)$, i.e., lattice paths with steps $(1,1)$ and $(1,-1)$ that never fall below the x-axis



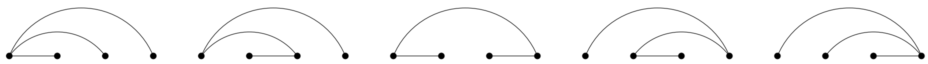
160. [3–] (unordered) pairs of lattice paths with $n+1$ steps each, starting at $(0,0)$, using steps $(1,0)$ or $(0,1)$, ending at the same point, and only intersecting at the beginning and end



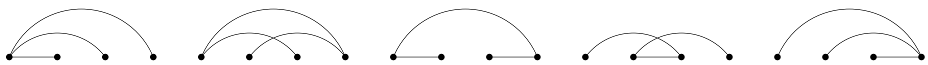
161. [2–] n nonintersecting chords joining $2n$ points on the circumference of a circle



162. [2] ways of drawing in the plane $n+1$ points lying on a horizontal line L and n arcs connecting them such that (α) the arcs do not pass below L, (β) the graph thus formed is a tree, (γ) no two arcs intersect in their interiors (i.e., the arcs are noncrossing), and (δ) at every vertex, all the arcs exit in the same direction (left or right)



163. [3–] ways of drawing in the plane $n+1$ points lying on a horizontal line L and n arcs connecting them such that (α) the arcs do not pass below L, (β) the graph thus formed is a tree, (γ) no arc (including its endpoints) lies strictly below another arc, and (δ) at every vertex, all the arcs exit in the same direction (left or right)



164. [3–] sequences of n 1's and n -1's such that every partial sum is nonnegative (with -1 denoted simply as $-$ below) (difficulty rating based on showing bijectively that the number of such sequences is $\frac{1}{n+1}\binom{2n}{n}$)

111––– 11–1–– 11––1– 1–11–– 1–1–1–

165. [1] sequences $1 \leq a_1 \leq \cdots \leq a_n$ of integers with $a_i \leq i$

111 112 113 122 123

166. [2] sequences $a_1, a_2, \ldots, a_n$ of integers such that $a_1 = 0$ and $0 \leq a_{i+1} \leq a_i + 1$

000 001 010 011 012

167. [1+] sequences $a_1, a_2, \ldots, a_{n-1}$ of integers such that $a_i \leq 1$ and all partial sums are nonnegative

$0,0 \quad 0,1 \quad 1,-1 \quad 1,0 \quad 1,1$

168. [2–] sequences $a_1, a_2, \ldots, a_n$ of integers such that $a_i \geq -1$, all partial sums are nonnegative, and $a_1 + a_2 + \cdots + a_n = 0$

$0,0,0 \quad 0,1,-1 \quad 1,0,-1 \quad 1,-1,0 \quad 2,-1,-1$

169. [2–] Sequences of $n-1$ 1's and any number of -1's such that every partial sum is nonnegative

$$1,1 \qquad 1,1,-1 \qquad 1,-1,1 \qquad 1,1,-1,-1 \qquad 1,-1,1,-1$$

170. [3–] Sequences $a_1 a_2 \cdots a_n$ of nonnegative integers such that $a_j = \#\{i : i < j, a_i < a_j\}$ for $1 \le j \le n$

$$000 \qquad 002 \qquad 010 \qquad 011 \qquad 012$$

171. [2+] Pairs (α, β) of compositions of n with the same number of parts, such that $\alpha \ge \beta$ (dominance order, i.e., $\alpha_1 + \cdots + \alpha_i \ge \beta_1 + \cdots + \beta_i$ for all i)

$$(111,111) \quad (12,12) \quad (21,21) \quad (21,12) \quad (3,3)$$

172. [2] permutations $a_1 a_2 \cdots a_{2n}$ of the multiset $\{1^2, 2^2, \ldots, n^2\}$ such that: (i) the first occurrences of $1, 2, \ldots, n$ appear in increasing order, and (ii) there is no subsequence of the form $\alpha\beta\alpha\beta$

$$112233 \quad 112332 \quad 122331 \quad 123321 \quad 122133$$

173. [3–] permutations $a_1 a_2 \cdots a_n$ of $[n]$ with longest decreasing subsequence of length at most two (i.e., there does not exist $i < j < k$, $a_i > a_j > a_k$), called 321-*avoiding* permutations

$$123 \quad 213 \quad 132 \quad 312 \quad 231$$

174. [2] permutations $a_1 a_2 \cdots a_n$ of $[n]$ for which there does not exist $i < j < k$ and $a_j < a_k < a_i$ (called 312-*avoiding* permutations)

$$123 \quad 132 \quad 213 \quad 231 \quad 321$$

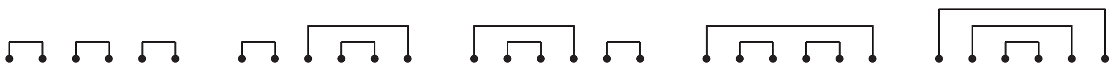
175. [2] permutations w of $[2n]$ with n cycles of length two, such that the product $(1, 2, \ldots, 2n) \cdot w$ has $n+1$ cycles

$$\begin{aligned}
(1,2,3,4,5,6)(1,2)(3,4)(5,6) &= (1)(2,4,6)(3)(5)\\
(1,2,3,4,5,6)(1,2)(3,6)(4,5) &= (1)(2,6)(3,5)(4)\\
(1,2,3,4,5,6)(1,4)(2,3)(5,6) &= (1,3)(2)(4,6)(5)\\
(1,2,3,4,5,6)(1,6)(2,3)(4,5) &= (1,3,5)(2)(4)(6)\\
(1,2,3,4,5,6)(1,6)(2,5)(3,4) &= (1,5)(2,4)(3)(6)
\end{aligned}$$

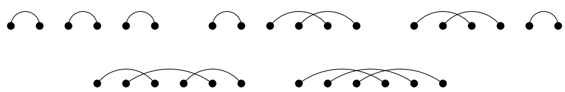
176. [3–] pairs (u, v) of permutations of $[n]$ such that u and v have a total of $n+1$ cycles, and $uv = (1, 2, \ldots, n)$

$$(1)(2)(3) \cdot (1,2,3) \quad (1,2,3) \cdot (1)(2)(3) \quad (1,2)(3) \cdot (1,3)(2)$$
$$(1,3)(2) \cdot (1)(2,3) \quad (1)(2,3) \cdot (1,2)(3)$$

177. [1+] *noncrossing matchings* of $[2n]$, i.e., ways of connecting $2n$ points in the plane lying on a horizontal line by n nonintersecting arcs, each arc connecting two of the points and lying above the points



178. [2+] *nonnesting matchings* on $[2n]$, i.e., ways of connecting $2n$ points in the plane lying on a horizontal line by n arcs, each arc connecting two of the points and lying above the points, such that no arc is contained entirely below another



179. [2] *noncrossing partitions* of $[n]$, i.e., partitions of $[n]$ such that if a, c appear in a block B and b, d appear in a block B', where $a < b < c < d$, then $B = B'$

123 12–3 13–2 23–1 1–2–3

(The unique partition of [4] that isn't noncrossing is 13–24.)

180. [3–] noncrossing partitions of $[2n+1]$ into $n+1$ blocks, such that no block contains two consecutive integers

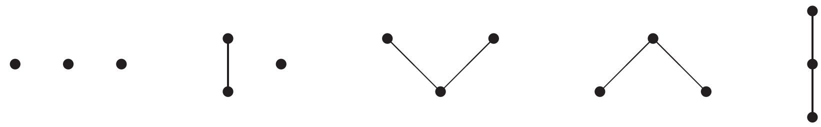
137–46–2–5 1357–2–4–6 157–24–3–6 17–246–3–5 17–26–35–4

181. [3–] *nonnesting partitions* of $[n]$, i.e., partitions of $[n]$ such that if a, e appear in a block B and b, d appear in a *different* block B' where $a < b < d < e$, then there is a $c \in B$ satisfying $b < c < d$

123 12–3 13–2 23–1 1–2–3

(The unique partition of [4] that isn't nonnesting is 14–23.)

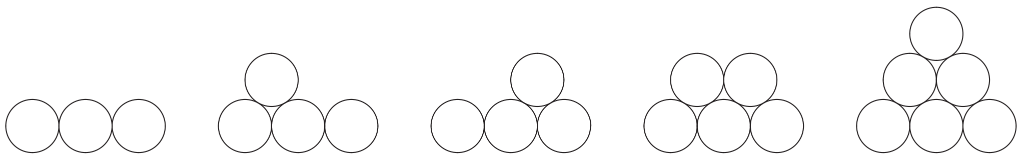
182. [3–] nonisomorphic n-element posets (i.e., partially ordered sets) with no induced subposet isomorphic to $\mathbf{2}+\mathbf{2}$ or $\mathbf{3}+\mathbf{1}$, where $\mathbf{a}+\mathbf{b}$ denotes the disjoint union of an a-element chain and a b-element chain



183. [2+] relations R on $[n]$ that are reflexive (iRi), symmetric ($iRj \Rightarrow jRi$), and such that if $1 \le i < j < k \le n$ and iRk, then iRj and jRk (in the example below we write ij for the pair (i, j), and we omit the pairs ii)

$$\emptyset \quad \{12, 21\} \quad \{23, 32\} \quad \{12, 21, 23, 32\} \quad \{12, 21, 13, 31, 23, 32\}$$

184. [2–] ways to stack coins in the plane, the bottom row consisting of n consecutive coins



185. [3–] n-tuples $(a_1, a_2, \ldots, a_n)$ of integers $a_i \ge 2$ such that in the sequence $1a_1a_2 \cdots a_n1$, each a_i divides the sum of its two neighbors

14321 13521 13231 12531 12341

186. [3] n-element subsets S of $\mathbb{N} \times \mathbb{N}$ such that if $(i, j) \in S$ then $i \ge j$ and there is a lattice path from $(0, 0)$ to (i, j) with steps $(0, 1)$, $(1, 0)$, and $(1, 1)$, such that all vertices of L lie in S

$$\{(0,0),(1,0),(2,0)\} \quad \{(0,0),(1,0),(1,1)\} \quad \{(0,0),(1,0),(2,1)\}$$

$$\{(0,0),(1,1),(2,1)\} \quad \{(0,0),(1,1),(2,2)\}$$

$$\begin{array}{c}
1\quad 1\quad 1\quad 1\quad 1\quad 1\quad 1\quad 1\quad 1\quad 1\quad 1\quad 1\quad 1\quad 1\\
1\quad 3\quad 2\quad 1\quad 5\quad 1\quad 2\quad 3\quad 1\quad 3\quad 2\quad 1\quad 5\\
2\quad 5\quad 1\quad 4\quad 4\quad 1\quad 5\quad 2\quad 2\quad 5\quad 1\quad 4\\
3\quad 2\quad 3\quad 3\quad 3\quad 2\quad 3\quad 3\quad 3\quad 2\quad 3\\
1\quad 5\quad 2\quad 2\quad 5\quad 1\quad 4\quad 4\quad 1\quad 5\\
2\quad 3\quad 1\quad 3\quad 2\quad 1\quad 5\quad 1\quad 2\\
1\quad 1\quad 1\quad 1\quad 1\quad 1\quad 1\quad 1
\end{array}$$

Figure 1: The frieze pattern corresponding to the sequence $(1,3,2,1,5,1,2,3)$

187. [3] positive integer sequences $a_1, a_2, \ldots, a_{n+2}$ for which there exists an integer array (called a *frieze pattern*, necessarily with $n+1$ rows)

$$\begin{array}{c}
1\quad\ 1\quad\ 1\quad\ \cdots\quad\ 1\quad\ 1\quad\ 1\quad\ \cdots\quad\ 1\quad\ 1\\
a_1\quad a_2\quad a_3\quad \cdots\quad a_{n+2}\quad a_1\quad a_2\quad \cdots\quad a_{n-1}\\
b_1\quad b_2\quad b_3\quad \cdots\quad b_{n+2}\quad b_1\quad \cdots\quad b_{n-2}\\
\vdots\\
r_1\quad r_2\quad r_3\quad \cdots\quad r_{n+2}\quad r_1\\
1\quad 1\quad 1\quad \cdots\quad 1
\end{array} \tag{8}$$

such that any four neighboring entries in the configuration $\begin{smallmatrix} & r & \\ s & & t \\ & u & \end{smallmatrix}$ satisfy $st = ru + 1$ (an example of such an array for $(a_1, \ldots, a_8) = (1,3,2,1,5,1,2,3)$ (necessarily unique) is given by Figure 1):

12213 22131 21312 13122 31221

188. [3] n-tuples $(a_1, \ldots a_n)$ of positive integers such that the tridiagonal matrix

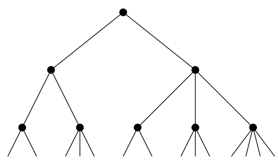
$$\left[\begin{array}{cccccccc}
a_1 & 1 & 0 & 0 & \cdots & & 0 & 0\\
1 & a_2 & 1 & 0 & \cdots & & 0 & 0\\
0 & 1 & a_3 & 1 & \cdots & & 0 & 0\\
 & & & & \vdots & & & \\
0 & 0 & 0 & 0 & \cdots & & a_{n-1} & 1\\
0 & 0 & 0 & 0 & \cdots & & 1 & a_n
\end{array}\right]$$

is positive definite with determinant one

NOTE. A real matrix A is *positive definite* if it is symmetric and every eigenvalue is positive; equivalently, A is symmetric and every leading principal minor is positive. A *leading principal minor* is the determinant of a square submatrix that fits into the upper left-hand corner of A.

$$131 \quad 122 \quad 221 \quad 213 \quad 312$$

189. [2+] Vertices of height $n-1$ of the tree T defined by the property that the root has degree 2, and if the vertex x has degree k, then the children of x have degrees $2, 3, \ldots, k+1$



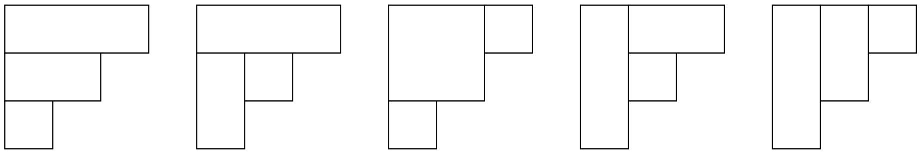
190. [3–] Subsets S of $\mathbb{N}$ such that $0 \in S$ and such that if $i \in S$ then $i+n, i+n+1 \in S$

$$\mathbb{N}, \quad \mathbb{N}-\{1\}, \quad \mathbb{N}-\{2\}, \quad \mathbb{N}-\{1,2\}, \quad \mathbb{N}-\{1,2,5\}$$

191. [2+] Ways to write $(1, 1, \ldots, 1, -n) \in \mathbb{Z}^{n+1}$ as a sum of vectors $e_i - e_{i+1}$ and $e_j - e_{n+1}$, without regard to order, where e_k is the kth unit coordinate vector in $\mathbb{Z}^{n+1}$:

$$\begin{gathered}
(1,-1,0,0)+2(0,1,-1,0)+3(0,0,1,-1)\\
(1,0,0,-1)+(0,1,-1,0)+2(0,0,1,-1)\\
(1,-1,0,0)+(0,1,-1,0)+(0,1,0,-1)+2(0,0,1,-1)\\
(1,-1,0,0)+2(0,1,0,-1)+(0,0,1,-1)\\
(1,0,0,-1)+(0,1,0,-1)+(0,0,1,-1)
\end{gathered}$$

192. [1+] tilings of the staircase shape $(n, n-1, \ldots, 1)$ with n rectangles such that each rectangle contains a square at the end of some row



193. [2+] $n \times n$ $\mathbb{N}$-matrices $M = (m_{ij})$ where $m_{ij} = 0$ unless $i = n$ or $i = j$ or $i = j - 1$, with row and column sum vector $(1, 2, \dots, n)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

This concludes the list of objects counted by Catalan numbers. A few more problems related to Catalan numbers are the following.

194. [*] We have
$$\sum_{k=0}^{n} C_{2k} C_{2(n-k)} = 4^n C_n.$$

195. [*] An intriguing variation of Problem 193 above is the following. A bijective proof would be of great interest. Let $g(n)$ denote the number of $n \times n$ $\mathbb{N}$-matrices $M = (m_{ij})$ where $m_{ij} = 0$ if $i > j + 1$, with row and column sum vector $\left(1, 3, 6, \dots, \binom{n+1}{2}\right)$. For instance, when $n = 2$ there are the two matrices
$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$
Then $g(n) = C_1 C_2 \cdots C_n$.

196. [2+] (compare with Problem 191) Let $f(n)$ be the number of ways to write the vector
$$\left(1, 2, 3, \dots, n, -\binom{n+1}{2}\right) \in \mathbb{Z}^{n+1}$$
as a sum of vectors $e_i - e_j$, $1 \le i < j \le n+1$, without regard to order, where e_k is the kth unit coordinate vector in $\mathbb{Z}^{n+1}$. For instance, when $n = 2$ there are the two ways $(1, 0, -1) + 2(0, 1, -1) = (1, -1, 0) + 3(0, 1, -1)$. Assuming Problem 195, show that $f(n) = C_1 C_2 \cdots C_n$.

197. [3–] The *Narayana numbers* $N(n,k)$ are defined by

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Let X_{nk} be the set of all sequences $w = w_1 w_2 \cdots w_{2n}$ of n 1's and n -1's with all partial sums nonnegative, such that

$$k = \#\{j : w_j = 1, w_{j+1} = -1\}.$$

Show that $N(n,k) = \#X_{nk}$. Hence by Problem 164, there follows

$$\sum_{k=1}^{n} N(n,k) = C_n.$$

One therefore says that the Narayana numbers are a *refinement* of the Catalan numbers. There are many other interesting refinements of Catalan numbers, but we won't consider them here.

198. [*] Let $f(n,k)$ be the number of ways to draw $k(n-2k-1)$ edges (the maximum possible) between vertices of a convex n-gon P so that (a) the vertices of each edge are at distance at least $k+1$ apart (where the distance between vertices u and v is the minimum number of steps from u to v along the edges of P), and (b) there do not exist $k+1$ edges such that any two of them intersect in their interiors. For instance, $f(n+2,1) = C_n$ by Problem 153. Then $f(n,k)$ is equal to the number of k-tuples $(D_1,\dots,D_k)$ of Dyck paths (as defined in Problem 159) from $(0,0)$ to $(2n-4k,0)$ such that D_i never rises above D_{i-1} for $1 < i \le k$.

NOTE. It can be shown that

$$\begin{aligned} f(n,k) &= \det[C_{n-i-j}]_{i,j=1}^{k} \\ &= \prod_{1 \le i < j \le n-2k} \frac{2k+i+j-1}{i+j-1}. \end{aligned}$$

199. [*] An *Eulerian tour* in a directed graph D is a permutation $e_1 e_2 \cdots e_q$ of the edges of D such that the final vertex (head) of e_i is the intial vertex (tail) of e_{i+1}, $1 \le i \le q$, where the subscripts are taken modulo q. Thus any cyclic shift $e_i e_{i+1} \cdots e_q e_1 \cdots e_{i-1}$ of an Eulerian tour is

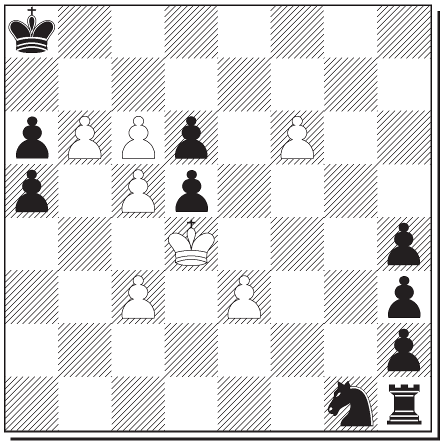
also an Eulerian tour. For $n \geq 2$, the number of loopless (i.e., no edge from a vertex to itself) digraphs on the vertex set $[n]$ with no isolated vertices and with exactly one Eulerian tour (up to cyclic shift) is given by $\frac{1}{2}(n-1)!C_n = (2n-1)_{n-2}$.

Bonus Chess Problem

(related to Problem 164)

R. Stanley (after E. Bonsdorff and K. Väisänen)

2003



Serieshelpmate in 34: how many solutions?

6. Young Tableaux

Let $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$. A *standard Young tableau* (SYT) of shape λ is a left-justified array of the integers $1, 2, \dots, n$, each occurring exactly once, with λ_i entries in the ith row, such that every row and column is increasing. An example of an SYT of shape $(4, 4, 2)$ is given by

$$\begin{array}{llll} 1 & 2 & 3 & 6 \\ 4 & 5 & 8 & 10 \\ 7 & 9 & & \end{array} .$$

We write f^λ for the number of SYT of shape λ.

Let u be a square of the Young diagram of λ, denoted $u \in \lambda$. The *hook length* $h(u)$ of u is the number of squares directly to the right or directly below u, counting u itself once. If $u = (i, j)$ (i.e., u is in the ith row and jth column of (the Young diagram of) λ), then $h(u) = \lambda_i + \lambda'_j - i - j + 1$. The hook lengths of $(4, 4, 2)$ are given by

6	5	3	2
5	4	2	1
2	1		

The difficulty ratings in this section assume knowledge of the definition of the RSK algorithm but none of its deeper properties.

200. [1] The number of SYT of shape (n, n) is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

201. [3] The number of SYT of shape λ is given by

$$f^\lambda = \frac{n!}{\prod_{u \in \lambda} h(u)}.$$

This is the famous *hook-length formula* of Frame, Robinson, and Thrall (1954). It was only given a "satisfactory" bijective proof in 1997.

202. generalize $p(0) + \cdots + p(n)$

203. [2–] Show that $\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$. In other words, the number of pairs (P, Q) of SYT of the same shape and with n entries is $n!$.

204. [3] The total number of SYT with n entries is equal to the number of involutions $w \in \mathfrak{S}_n$, i.e., $w^2 = 1$.

205. [3] The number of SYT with $2n$ entries and all rows of even length is $1 \cdot 3 \cdot 5 \cdots (2n-1)$.

206. [2] The number of SYT with n entries and at most two rows is $\binom{n}{\lfloor n/2 \rfloor}$.

207. [3] The number of SYT with n entries and at most three rows is equal to $\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} C_i$, where C_i denotes a Catalan number.

208. [3] The number of SYT with n entries and at most four rows is equal to $C_{\lfloor (n+1)/2 \rfloor} C_{\lceil (n+1)/2 \rceil}$.

    NOTE. There is a similar, though somewhat more complicated, formula for the case of five rows. For six and more rows, no "reasonable" formula is known.

209. [2] The number of pairs (P, Q) of SYT of the same shape with n entries each and at most two rows is the Catalan number C_n.

210. [3] The number of pairs (P, Q) of SYT of the same shape with n entries each and at most three rows is given by

$$\frac{1}{(n+1)^2(n+2)} \sum_{k=0}^{n} \binom{2k}{k} \binom{n+1}{k+1} \binom{n+2}{k+1}.$$

211. [2] Let $W_i(n)$ be the number of ways to draw i diagonals in a convex n-gon such that no two diagonals intersect in their interiors. Then $W_i(n)$ is the number of standard Young tableaux of shape $\langle (i+1)^2, 1^{n-i-3} \rangle$ (i.e., two parts equal to $i+1$ and $n-i-3$ parts equal to 1; when $i = 0$ there are $n-1$ parts equal to 1).

    NOTE. Given the result of this problem, it follows immediately from the hook-length formula (Problem 201) that

$$W_i(n) = \frac{1}{n+i} \binom{n+i}{i+1} \binom{n-3}{i},$$

a result originally stated by Kirkman (1857) and Prouhet (1866), with the first complete proof by Cayley (1890-91).

212. [2+] Let T be an SYT of shape $\lambda \vdash n$. For each entry of T not in the first column, let $f(i)$ be the number of entries j in the column immediately to the left of i and in a row not above i, for which $j < i$. Define $f(T) = \prod_i f(i)$, where i ranges over all entries of T not in the first column. For instance, if

$$T = \begin{array}{llll} 1 & 3 & 6 & 8 \\ 2 & 4 & 7 & \\ 5 & & & \end{array} ,$$

then $f(3) = 2$, $f(4) = 1$, $f(6) = 2$, $f(7) = 1$, $f(8) = 2$, and $f(T) = 8$. Then $\sum_{\mathrm{sh}(T)=\lambda} f(T)$, where the sum ranges over all SYT T of shape λ, is equal to the number of partitions of the set $[n]$ of type λ (i.e., with block sizes $\lambda_1, \lambda_2, \dots$).

213. [3+] Let $\lambda \vdash n$. An assignment $u \mapsto a_u$ of the distinct integers $1, 2, \dots, n$ to the squares $u \in \lambda$ is a *balanced tableau* of shape λ if for each $u \in \lambda$ the number a_u is the kth largest number in the hook of u, where k is the leg-length (number of squares directly below u, counting u itself) of the hook of u. For instance, the balanced tableaux of shape $(3, 2)$ are

$$\begin{array}{lll} 4 & 2 & 1 \\ 5 & 3 & \end{array} \quad \begin{array}{lll} 4 & 2 & 3 \\ 5 & 1 & \end{array} \quad \begin{array}{lll} 4 & 2 & 5 \\ 3 & 1 & \end{array} \quad \begin{array}{lll} 4 & 3 & 5 \\ 2 & 1 & \end{array} \quad \begin{array}{lll} 3 & 2 & 1 \\ 5 & 4 & \end{array} .$$

Let b^λ be the number of balanced tableaux of shape λ. Then $b^\lambda = f^\lambda$, the number of SYT of shape λ.

NOTE. For such a simply stated problem, this seems remarkably difficult to prove.

214. [2] Let $f(n)$ be the number of ways to write the permutation $n, n-1, n-2, \dots, 1 \in \mathfrak{S}_n$ as a product of $\binom{n}{2}$ (the minimum possible) adjacent transpositions $s_i = (i, i+1)$, $1 \le i \le n-1$. For instance, $f(3) = 2$, corresponding to $s_1 s_2 s_1$ and $s_2 s_1 s_2$. Then $f(n)$ is equal to the number of balanced tableaux (as defined in the previous problem) of shape $(n-1, n-2, \dots, 1)$.

NOTE. It thus follows from the previous problem that $f(n) = f^{(n-1,n-2,\dots,1)}$. *Any* bijective proof of this difficult result would be an impressive achievement.

215. [3+] Let $w_0 = n, n-1, n-2, \dots, 1 \in \mathfrak{S}_n$ and $p = \binom{n}{2}$. Define

$$R_n = \{(a_1, \dots, a_p) \in [n-1]^p \,:\, w_0 = s_{a_1} s_{a_2} \cdots s_{a_p}\},$$

where $s_i = (i, i+1)$ as in the previous problem. For example, $R_3 = \{(1,2,1), (2,1,2)\}$. Then

$$\sum_{(a_1,\dots,a_p)\in R_n} a_1 a_2 \cdots a_p = p!. \tag{9}$$

For instance, when $n = 3$ we get $1 \cdot 2 \cdot 1 + 2 \cdot 1 \cdot 2 = 3!$.

UPDATE (May 19, 2015). The difficulty rating has been changed from [*] to [3+] due to the paper by Benjamin Young at `arxiv.org/abs/1409.7714`.

216. [*] Let n, p, and w_0 be as is the previous problem. Let T_n be the set of all sequences $((i_1, j_1), \dots, (i_p, j_p))$ such that

- $1 \le i_k < j_k \le n$ for $1 \le k \le p$
- $w_0 = (i_1, j_1)(i_2, j_2) \cdots (i_p, j_p)$, where (i_k, j_k) denotes the transposition exchanging i_k and j_k
- Let $v_k = (i_1, j_1)(i_2, j_2) \cdots (i_k, j_k)$. Then for all $1 \le k < p$, we have $\mathrm{inv}(v_{k+1}) = 1 + \mathrm{inv}(v_k)$, where inv is defined in Problem 45.

Then

$$\sum_{((i_1,j_1),\dots,(i_p,j_p))\in T_n} (j_1 - i_1)(j_2 - i_2) \cdots (j_p - i_p) = p!. \tag{10}$$

For instance, $T_3 = \{((1,2),(2,3),(1,2)), ((1,2),(1,3),(2,3)),$ $((2,3),(1,2),(2,3)), ((2,3),(1,3)(1,2))\}$, giving $1 \cdot 1 \cdot 1 + 1 \cdot 2 \cdot 1 + 1 \cdot 1 \cdot 1 + 1 \cdot 2 \cdot 1 = 3!$.

NOTE. Note the similarity with the previous problem. Is it just a coincidence that the two sums (9) and (10) are equal?

217. [*] Continuing the notation of Problem 215, if $\alpha = (a_1, a_2, \dots, a_p) \in R_n$, then define

$$f(\alpha) = \#\{i \,:\, 1 \le i \le p-2,\ a_i = a_{i+2}\}.$$

Then $\sum_{\alpha \in R_n} f(\alpha) = \#R_n$ $(= f^{(n-1,n-2,\dots,1)}$ by Problem 214).

218. [3–] An *oscillating tableau* of length $2n$ and shape $\emptyset$ is a sequence

$$(\lambda^0, \lambda^1, \ldots, \lambda^{2n}),$$

where each λ^i is a partition, $\lambda^0 = \lambda^{2n} = \emptyset$, and each λ^i is obtained from λ^{i-1} by either adding or removing a square from (the diagram of) λ. For instance, when $n = 2$ we get the three oscillating tableaux $(\emptyset, 1, \emptyset, 1, \emptyset)$, $(\emptyset, 1, 2, 1, \emptyset)$, and $(\emptyset, 1, 11, 1, \emptyset)$. The number of oscillating tableaux of length $2n$ and shape $\emptyset$ is equal to $1 \cdot 3 \cdot 5 \cdots (2n-1)$ (the number of partitions of $[2n]$ into n 2-element blocks).

219. [*] The *major index* $\mathrm{maj}(T)$ of an SYT is defined to be the sum of all entries i of T for which $i+1$ appears in a lower row than i. Fix $n \in \mathbb{P}$ and $\lambda \vdash n$, and let $m \in \mathbb{Z}$. Then the number of SYT T of shape λ satisfying $\mathrm{maj}(T) \equiv m \pmod{n}$ depends only on λ and $\gcd(m, n)$.

220. [3–] Let μ be a partition, and let A_μ be the infinite shape consisting of the quadrant $Q = \{(i, j) : i < 0,\ j > 0\}$ with the shape μ removed from the lower right-hand corner. Thus every square of A_μ has a finite hook and hence a hook length. For instance, when $\mu = (3, 1)$ we get the diagram

$$\begin{array}{c|c|c|c|c|c|c|}
 & & & & \vdots & & \\
\hline
 & 10 & 9 & 8 & 6 & 5 & 3 \\
\hline
 & 9 & 8 & 7 & 5 & 4 & 2 \\
\hline
\cdots & 8 & 7 & 6 & 4 & 3 & 1 \\
\hline
 & 6 & 5 & 4 & 2 & 1 \\
\cline{1-6}
 & 3 & 2 & 1 \\
\cline{1-4}
\end{array}$$

Then the multiset of hook lengths of A_μ is equal to the union of the multiset of hook lengths of Q (explicitly given by $\{1^1, 2^2, 3^3, \ldots\}$) and the multiset of hook lengths of μ.

221. [3–] A *plane partition* of n is an array $\pi = (\pi_{ij})_{i,j\geq 1}$ of nonnegative integers whose rows and columns are weakly decreasing and whose entries sum to n. When writing π, the entries equal to 0 are often omitted. Thus the plane partitions of the integers $0 \leq n \leq 3$ are given by

$$\emptyset \quad 1 \quad 2 \quad 11 \quad \begin{matrix}1\\1\end{matrix} \quad 3 \quad 21 \quad 111 \quad \begin{matrix}11\\1\end{matrix} \quad \begin{matrix}2\\1\end{matrix} \quad \begin{matrix}1\\1\\1.\end{matrix}$$

If π is a plane partition of n, then we write $|\pi| = n$. Let $a_{rs}(n)$ denote the number of plane partitions of n with at most r rows and at most s columns (of nonzero entries). Then

$$\sum_{n\geq 0} a_{rs}(n)x^n = \prod_{i=1}^{r}\prod_{j=1}^{s}(1-x^{i+j-1})^{-1}. \tag{11}$$

In particular, let $a(n)$ denote the total number of plane partitions of n. If we let $r, s \to \infty$ in (11) then it's not hard to see that we get

$$\sum_{n\geq 0} a(n)x^n = \prod_{i\geq 1}(1-x^i)^{-i},$$

a famous formula of MacMahon.

Hint. Use the RSK algorithm.

Note. At this point it's natural to consider *three-dimensional* (and higher) partitions, but almost nothing is known about them, and a "reasonable" enumeration of them is believed to be hopeless.

222. [3+] Fix $r, s, t > 0$. Let $\mathcal{P}(r,s,t)$ denote the set of plane partitions with at most r rows, at most s columns, and with largest part at most t. Then

$$\sum_{\pi\in\mathcal{P}(r,s,t)} x^{|\pi|} = \prod_{i=1}^{r}\prod_{j=1}^{s}\prod_{k=1}^{t}\frac{1-x^{i+j+k-1}}{1-x^{i+j+k-2}}. \tag{12}$$

Note that Problem 221 is the case $t \to \infty$.

223. [3] A plane partition $\pi = (\pi_{ij})$ is *symmetric* if $\pi_{ij} = \pi_{ji}$ for all i, j. Let $b(n)$ denote the number of symmetric plane partitions of n. Then

$$\sum_{n\geq 0} b(n)x^n = \prod_{i\geq 1}\frac{1}{(1-x^{2i-1})\,(1-x^{2i})^{\lfloor i/2\rfloor}}.$$

224. [2] Let $f_{rs}(n)$ denote the number of plane partitions $\pi = (\pi_{ij})$ with at most r rows, at most s columns, and with *trace* $\mathrm{tr}(\pi) := \pi_{11}+\pi_{22}+\cdots = n$. Then

$$f_{rs}(n) = \binom{rs+n-1}{rs-1}.$$

225. [2] A *monotone triangle* of length n is a triangular array of integers whose first row is $1, 2, \ldots, n$, every row is strictly increasing, and each entry is (weakly) between its two neighbors above. This somewhat vague definition should be made clear by the following example:

$$\begin{array}{ccccccccccc}
1 & & 2 & & 3 & & 4 & & 5 & & 6 \\
& 1 & & 2 & & 3 & & 4 & & 6 & \\
& & 1 & & 3 & & 4 & & 5 & & \\
& & & 2 & & 4 & & 5 & & & \\
& & & & 2 & & 5 & & & & \\
& & & & & 3 & & & & &
\end{array}.$$

There are for instance seven monotone triangles of length 3, given by

$$\begin{array}{c}123\\12\\1\end{array}\quad\begin{array}{c}123\\12\\2\end{array}\quad\begin{array}{c}123\\13\\1\end{array}\quad\begin{array}{c}123\\13\\2\end{array}\quad\begin{array}{c}123\\13\\3\end{array}\quad\begin{array}{c}123\\23\\2\end{array}\quad\begin{array}{c}123\\23\\3\end{array}.$$

An *alternating sign matrix* is a square matrix with entries $0, \pm 1$, such that the nonzero entries in every row and column alternate $1, -1, 1, -1, \ldots, 1, -1, 1$. (Thus every row and column sum is 1.) An example is

$$\left[\begin{array}{cccccc}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 & 0 \\
1 & -1 & 1 & -1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{array}\right].$$

The number of monotone triangles of length n is equal to the number of $n \times n$ alternating sign matrices.

226. [*] An $n \times n$ *totally symmetric self-complementary (TSSC) plane partition* is a plane partition $\pi = (\pi_{ij})_{i,j=1}^{n}$ satisfying: (i) π is symmetric

(as defined in Problem 223), (ii) every row (and hence every column by symmetry) of π is a self-conjugate partition, and (iii) π is invariant under the operation of replacing each entry i with $n-i$ and rotating 180°. It is easy to see that n must be even. The 4×4 TSSC plane partitions are given by:

$$\begin{bmatrix} 4 & 4 & 2 & 2 \\ 4 & 4 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 4 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}.$$

A *descending plane partition* is an array of positive integers satisfying: (i) Each row after the first contains fewer elements than the row above, (ii) each row is indented one space to the right from the row above, (iii) the entries weakly decrease in each row, (iv) the entries strictly decrease in each column, (v) the first entry in each row (except the first) does not exceed the number of entries in the preceding row, and (vi) the first entry in each row is greater than the number of entries in its own row. The descending partitions with largest part at most three are given by

$$\emptyset \qquad 2 \qquad 3 \qquad 31 \qquad 32 \qquad 33 \qquad \begin{array}{l} 33 \\ \ \ 2 \end{array}.$$

The following four numbers are all equal:

(a) $\displaystyle\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$

(b) the number of monotone triangles of length n

(c) the number of $2n \times 2n$ TSSC plane partitions

(d) the number of descending plane partitions with largest part at most n.

Note. There are $\binom{4}{2} = 6$ pairs of equal numbers above. *None* of these six pairs is known to be equal by a bijective proof! (All are known to be equal by complicated indirect arguments.) This is one of the most intriguing open problems in the area of bijective proofs. There are certain refinements of the numbers (a)–(d) which may be useful in finding bijections. For instance, it appears that the number of descending plane partitions with largest part at most n and with exactly k parts equal to n is equal to the number of monotone triangles of length n and bottom element $k+1$. Similarly, it seems that the number of descending plane partitions with largest part at most n and with exactly k parts is equal to the number of monotone triangles of length n with exactly k entries which are greater than the entry to the upper left.

227. [3] If A is an alternating sign matrix, let $s(A)$ denote the number of -1's in A. Then

$$\sum_A 2^{s(A)} = 2^{\binom{n}{2}},$$

where A ranges over all $n \times n$ alternating sign matrices.

228. [3] Let $w = a_1 \cdots a_n \in \mathfrak{S}_n$. An *increasing subsequence* of w of length j is a subsequence $a_{i_1} a_{i_2} \cdots a_{i_j}$ of w (so $i_1 < i_2 < \cdots < i_j$) such that $a_{i_1} < a_{i_2} < \cdots < a_{i_j}$. *Decreasing subsequence* is defined analogously. Let $\mathrm{is}(w)$ (respectively, $\mathrm{ds}(w)$) denote the length of the longest increasing (respectively, decreasing) subsequence of w. A famous result of Erdős and Szekeres, given an equally famous elegant pigeonhole proof by Seidenberg, states that if $n = pq + 1$, then either $\mathrm{is}(w) > p$ or $\mathrm{ds}(w) > q$. The number $A(p,q)$ of $w \in \mathfrak{S}_{pq}$ satisfying $\mathrm{is}(w) = p$ and $\mathrm{ds}(w) = q$ is given by $(f^\lambda)^2$, where λ is the partition with p parts equal to q (i.e., the diagram of λ is a $p \times q$ rectangle). Note that the hook-length formula (Problem 201) then gives an explicit formula for $A(p,q)$.

229. [3] If T is an SYT with n entries, then let $w(T)$ be the permutation of $1, 2, \ldots, n$ obtained by reading the entries of T in the usual (English) reading order. For instance, if T is given by

$$\begin{array}{llll} 1 & 3 & 4 & 9 \\ 2 & 6 & 8 & \\ 5 & 7 & & \end{array} \quad ,$$

then $w(T) = 134926857 \in \mathfrak{S}_9$. Define

$$\mathrm{sgn}(T) = \begin{cases} 1, & \text{if } w(T) \text{ is an even permutation} \\ -1, & \text{if } w(T) \text{ is an odd permutation.} \end{cases}$$

Then

$$\sum_T \operatorname{sgn}(T) = 2^{\lfloor n/2 \rfloor},$$

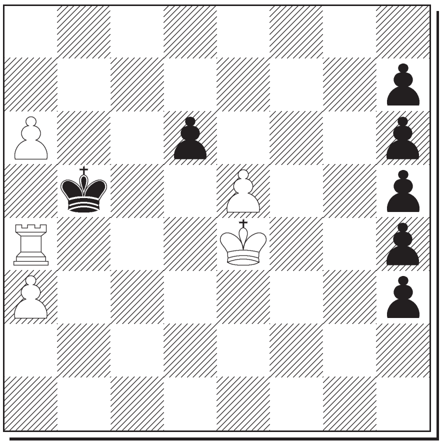
summed over all SYT with n entries.

Bonus Chess Problem

(related to Problem 201)

K. Väisänen

1985



Serieshelpmate in 25: how many solutions?

7. Lattice Paths and Tilings

Let S be a subset of $\mathbb{Z}^k$. A *lattice path* of length ℓ from $\alpha \in \mathbb{Z}^k$ to $\beta \in \mathbb{Z}^k$ with steps S may be regarded as a sequence

$$\alpha = v_0, v_1, \ldots, v_\ell = \beta$$

such that each $v_i - v_{i-1} \in S$. A number of lattice path problems have been given already: Problems 8, 158, 159, 160, and 186.

230. [2+] The number of lattice paths of length $2n$ from $(0,0)$ to $(0,0)$ with steps $(0,\pm 1)$ and $(\pm 1, 0)$ is $\binom{2n}{n}^2$.

231. [1] Let $f(m,n)$ denote the number of lattice paths from $(0,0)$ to (m,n) with steps $(0,1)$, $(1,0)$, $(1,1)$. Then

$$f(m+1,n+1) = f(m,n+1) + f(m+1,n) + f(m,n), \ \ m,n \geq 0.$$

232. [3] Continuing the previous problem, we have

$$(n+2)f(n+2,n+2) = 3(2n+3)f(n+1,n+1) - (n+1)f(n,n), \ \ n \geq 0.$$

233. [2] Let $1 \leq n < m$. The number of lattice paths from $(0,0)$ to (m,n) with steps $(1,0)$ and $(0,1)$ that intersect the line $y = x$ only at $(0,0)$ is given by $\frac{m-n}{m+n}\binom{m+n}{m}$.

    NOTE. There is an exceptionally elegant proof based on the formula

$$\frac{m-n}{m+n}\binom{m+n}{m} = \binom{m+n-1}{n} - \binom{m+n-1}{m}.$$

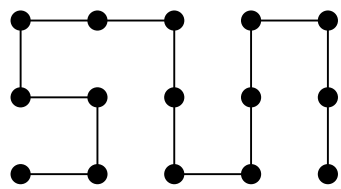
234. [2+] Let $S = \{(a,b) \ : \ a = 1,2,\ldots,n, \ \ b = 1,2,3\}$. A *rook tour* of S is a polygonal path made up of line segments connecting points $p_1, p_2, \ldots, p_{3n}$ in sequence such that

    (i) $p_i \in S$,

    (ii) p_i and p_{i+1} are a unit distance apart, for $1 \leq i < 3n$,

(iii) for each $p \in S$ there is a unique i such that $p_i = p$.

The number of rook tours that begin at $(1,1)$ and end at $(n,1)$ is 2^{n-2}. (An example of such a rook tour for $n=5$ is depicted below.)



235. [*] Let $f(m,n)$ denote the number of triples (α,β,γ), where α,β,γ are lattice paths from $(0,0)$ to (m,n) with steps $(1,0)$ and $(0,1)$, and where β and γ never rise above α. For instance, let $m=n=1$. If α is the path $(0,0),(0,1),(1,1)$, then there are 2^2 choices for (β,γ), while if α is the path $(0,0),(1,0),(1,1)$ there are 1^2 choices for (β,γ). Hence $f(1,1)=5$. In general,

$$f(m,n) = \frac{(m+n+1)!\,(2m+2n+1)!}{(m+1)!\,(2m+1)!\,(n+1)!\,(2n+1)!}.$$

236. Let $a_{i,j}(n)$ (respectively, $\bar{a}_{i,j}(n)$) denote the number of lattice paths of length n from $(0,0)$ to (i,j), with steps $(\pm1,0)$ and $(0,\pm1)$, never touching a point $(-k,0)$ with $k \geq 0$ (respectively, $k>0$) once leaving the starting point. Then:

   (a) [?] $a_{0,1}(2n+1) = 4^n C_n$

   (b) [3] $a_{1,0}(2n+1) = C_{2n+1}$

   (c) [?] $a_{-1,1}(2n) = \frac{1}{2}C_{2n}$

   (d) [?] $a_{1,1}(2n) = 4^{n-1}C_n + \frac{1}{2}C_{2n}$

   (e) [?] $\bar{a}_{0,0}(2n) = 2\cdot 4^n C_n - C_{2n+1}$.

237. Let $b_{i,j}(n)$ (respectively, $\bar{b}_{i,j}(n)$) denote the number of walks in n steps from $(0,0)$ to (i,j), with steps $(\pm1,\pm1)$, never touching a point $(-k,0)$ with $k \geq 0$ (respectively, $k>0$) once leaving the starting point. Then:

   (a) [3–] $b_{1,1}(2n+1) = C_{2n+1}$

(b) [?] $b_{-1,1}(2n+1) = 2 \cdot 4^n C_n - C_{2n+1}$

(c) [3–] $b_{0,2}(2n) = C_{2n}$

(d) [?] $b_{2i,0}(2n) = \frac{i}{n}\binom{2i}{i}\binom{2n}{n-i}4^{n-i}$, $i \geq 1$. (The case $i = 1$ has a known bijective proof.)

(e) [?] $\bar{b}_{0,0}(2n) = 4^n C_n$.

238. [*] The number of lattice paths with steps $(-1,0)$, $(0,-1)$, and $(1,1)$ from $(0,0)$ to $(i,0)$ of length $3n+2i$, and staying within the first quadrant (i.e., any point (a,b) along the path satisfies $a,b \geq 0$) is given by

$$\frac{4^n(2i+1)}{(n+i+1)(2n+2i+1)}\binom{2i}{i}\binom{3n+2i}{n}.$$

A (difficult) bijective proof is known for the case $i = 0$.

239. [3] Let $f(n)$ be the number of n-elements subsets S of $\mathbb{N} \times \mathbb{N}$ with the following properties.

(a) For some $k \geq 0$, $\{(0,k), (1,k-1), (2,k-2), \ldots, (k,0)\} \subseteq S$.

(b) Let k be as in (a). Then $(i,j) \in S \Rightarrow i + j \geq k$.

(c) Let k be as in (a). For any point $(i,j) \in S$, there is a lattice path L from some point (a,b) with $a + b = k$ to (i,j) with steps $(1,0)$ and $(0,1)$, such that all vertices of L lie in S.

Then $f(n) = 3^n$. (Problem 186 is similar.)

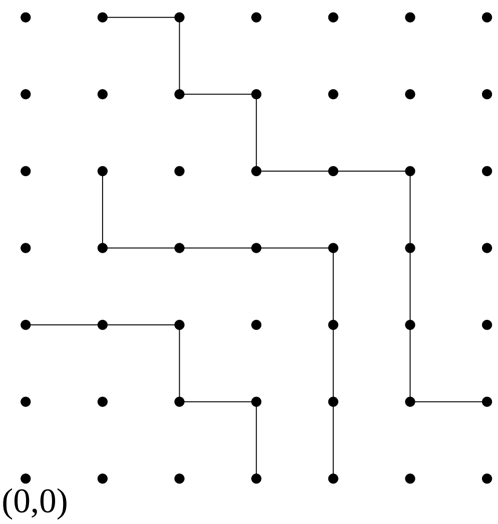
240. [*] Let $f_k(m,n)$ denote the number of ways a rook can move from a square S on an $m \times n$ chessboard back to S in k moves. Show that

$$f_k(m,n) =$$

$$\frac{(m+n-2)^k + (n-1)(m-2)^k + (m-1)(n-2)^k + (m-1)(n-1)(-2)^k}{mn}.$$

241. [3–] In this problem all lattice paths have steps $(1,0)$ and $(0,-1)$. An *n-path* is an n-tuple $\mathbf{L} = (L_1, \ldots, L_n)$ of lattice paths. Let $\alpha, \beta, \gamma, \delta \in \mathbb{N}^n$. We say that $\mathbf{L}$ is of *type* $(\alpha, \beta, \gamma, \delta)$ if L_i goes from (β_i, γ_i) to (α_i, δ_i). (Clearly then $\alpha_i \geq \beta_i$ and $\gamma_i \geq \delta_i$.) $\mathbf{L}$ is *intersecting* if for some $i \neq j$, L_i

and L_j have a point in common; otherwise $\mathbf{L}$ is *nonintersecting*. The diagram below illustrates a nonintersecting 3-path of type $((3,4,6), (0,1,1), (2,4,6), (0,0,1))$.



Let $B(\alpha,\beta,\gamma,\delta)$ be the number of nonintersecting n-paths of type $(\alpha,\beta,\gamma,\delta)$. Suppose that for any nonidentity permutation π of $1,2,\dots,n$, there does not exist a nonintersecting n-path whose paths go from (β_i,γ_i) to $(\alpha_{\pi(i)},\delta_{\pi(i)})$. (This is the case e.g. if $\alpha_1 < \cdots < \alpha_n$, $\beta_1 \le \cdots \le \beta_n$, $\gamma_1 < \cdots < \gamma_n$, and $\delta_1 \le \cdots \le \delta_n$.) Then

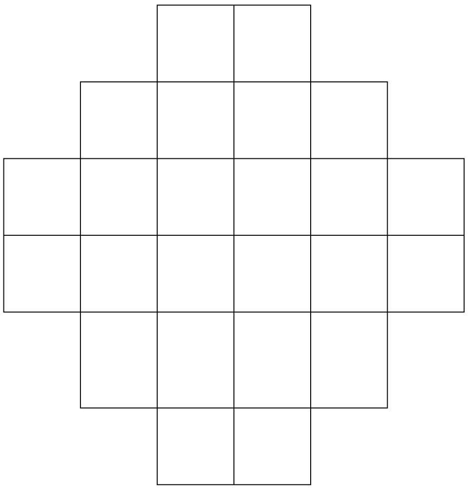
$$B(\alpha,\beta,\gamma,\delta) = \det\left[\binom{\alpha_j - \beta_i + \delta_j - \gamma_i}{\alpha_j - \beta_i}\right]_{i,j=1}^{n}.$$

242. [1+] The number of ways to tile a $2\times n$ board with n dominos (two edgewise adjacent squares, oriented either horizontally or vertically) is the Fibonacci number F_{n+1}.

243. [3] Given a finite sequence $\alpha = (2a_1, 2a_2, \dots, 2a_k)$ of positive even integers, let $B(\alpha)$ be the array of squares (or "board") consisting of $2a_i$ squares in the ith row (read top to bottom), with the centers of the rows lying on a vertical line. The *Aztec diamond* of order n is the board

$$\mathrm{AZ}_n = B(2,4,6,\dots,2n-4,2n-2,2n,2n,2n-2,2n-4,\dots,6,4,2).$$

For instance, AZ_3 looks like



The number of domino tilings of $AZ(n)$ is $2^{\binom{n+1}{2}}$.

244. [3–] The *augmented Aztec diamond* of order n is the board

$$AAZ_n = B(2,4,6,\ldots,2n-4,2n-2,2n,2n,2n,2n-2,2n-4,\ldots,6,4,2).$$

In other words, AAZ_n is obtained from AZ_n by adding a new row of length $2n$ in the middle. The number of domino tilings of AAZ_n is equal to the number $f(n)$ of lattice paths with steps $(1,0)$, $(0,1)$, and $(1,1)$ from $(0,0)$ to (n,n), as defined in Problem 230.

245. [3–] The *half Aztec diamond* of order n is the board

$$HAZ_n = B(2,4,6,\ldots,2n,2n).$$

The number of domino tilings of HAZ_n is the number of lattice paths with steps $(1,0)$, $(0,1)$, and $(1,1)$ from $(0,0)$ to (n,n) that never rise above the line $y=x$.

246. [2+] Let $r,s,t \in \mathbb{P}$. The number of tilings of a centrally-symmetric hexagon with side lengths r,s,t,r,s,t by rhombi with side lengths one is equal to the number of plane partitions with at most r rows, at most s columns, and with largest part at most t (given explicitly by equation (12)). Figure 2 shows an example with $(r,s,t)=(2,3,4)$.

HINT. Stare at Figure 2.

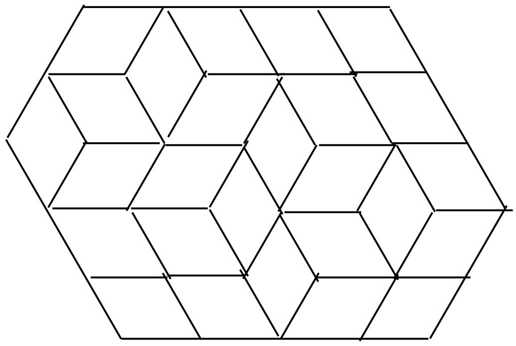


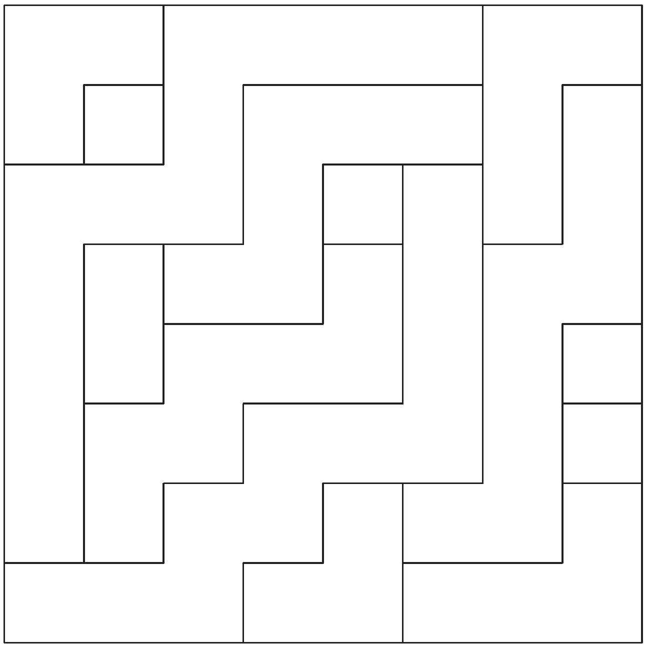
Figure 2: A centrally-symmetric hexagon tiled by rhombi

247. [*] The number $N(m,n)$ of domino tilings of a $2m \times 2n$ chessboard is given by

$$N(m,n) = 4^{mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left(\cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+1} \right).$$

Note. A combinatorial proof of this famous formula of Kasteleyn would consist of something like expanding the product, replacing the cosines with their expressions in terms of complex exponentials, finding a one-to-one correspondence between the chessboard tilings being counted and certain terms equal to 1 of the expanded product, and showing that the remaining terms all cancel out.

248. [3–] A *snake* on the $m \times n$ chessboard is a nonempty subset S of the squares of the board with the following property: Start at one of the squares and continue walking one step up or to the right, stopping at any time. The squares visited are the squares of the snake. Here is an example of the 8×8 chessboard covered with disjoint snakes.



The total number of ways to cover an $m \times n$ chessboard (and many other nonrectangular boards as well, such as the Young diagram of a partition) with disjoint snakes is a product of Fibonacci numbers.

249. [3] The *Somos-4 sequence* $a_0, a_1, \ldots$ is defined by

$$a_n a_{n+4} = a_{n+1} a_{n+3} + a_{n+2}^2, \quad n \geq 0, \tag{13}$$

with the initial conditions $a_0 = a_1 = a_2 = a_3 = 1$. Show that each a_n is an integer by finding a combinatorial interpretation of a_n and verifying combinatorially that (13) holds. The known combinatorial interpretation of a_n is as the number of matchings (vertex-disjoint sets of edges covering all the vertices) of certain graphs G_n.

NOTE. A similar interpretation is known for the terms of the *Somos-5 sequence*, defined by

$$a_n a_{n+5} = a_{n+1} a_{n+4} + a_{n+2} a_{n+3}, \quad n \geq 0,$$

with $a_0 = a_1 = a_2 = a_3 = a_4 = 1$. It is known the terms of the Somos-6 and Somos-7 sequences are integers, but no combinatorial proof (or simple proof in general) is known. These sequences are defined by

$$a_n a_{n+6} = a_{n+1} a_{n+5} + a_{n+2} a_{n+4} + a_{n+3}^2, \quad n \geq 0,$$

with $a_0 = \cdots = a_5 = 1$, and

$$a_n a_{n+7} = a_{n+1}a_{n+6} + a_{n+2}a_{n+5} + a_{n+3}a_{n+4}, \quad n \geq 0,$$

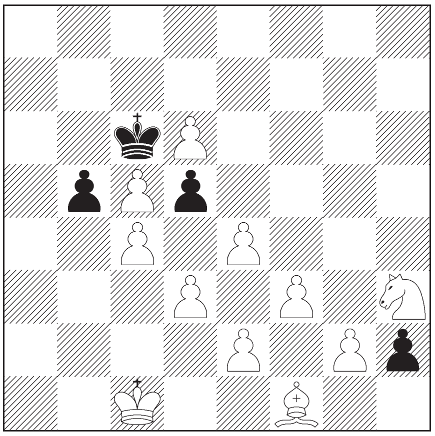
with $a_0 = \cdots = a_6 = 1$. By now it should be obvious what the definition of the *Somos-k* sequence is for any $k \geq 2$. Somewhat surprisingly, the terms of the Somos-8 sequence (and presumably Somos-k for all $k > 8$, though I'm not sure whether this is known) are not all integers; the first noninteger for Somos-8 is $a_{17} = 420514/7$.

Bonus Chess Problem

(related to Problem 235)

(C) R. Stanley, 3rd Prize

E. Bonsdorff 80th birthday tourney, 2002



Serieshelpmate in 14: how many solutions?

CHRONOLOGY OF NEW PROBLEMS (beginning 9/8/08)

151. September 8, 2008

91. September 11, 2008

58. August 18, 2009

59. August 18, 2009